SOLUTIONS OF THE EXERCISES

IN

MESSRS. HALL AND STEVENS',

NEW SCHOOL GEOMETRY

PART III

BY

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PREFACE

The present work provides full solutions of all the numerical, graphical and theoretical exercises on Messrs. Hall and Stevens' New School Geometry. Every effort is made in the work to make the student well grasp the subject of Geometry, and take a deep interest in its useful applications. It is for this latter purpose chiefly that there is given a diagram with almost every exercise, without which the subject is sure to become quite dry and tedious for the student, who finds no interest in the subject and is ultimately obliged to give up its study and depend upon other branches of Mathematics to make up its dificiency. In order to remove this difficulty, and excite the eagerness of young students and thus make them study the subject thoroughly. The diagrams are printed black which has undoubtedly made them appear far more beautiful than the ordinary diagrams.

In practical work, however, for the sake of convenience, the figures are drawn on a reduced scale, the representative fraction whereof has been given at the head of each figure. By the term *representaivet fraction of a scale* is meant the ratio of the

scale in which the figure is drawn to the original length. This fraction is, for the sake of abbreviation, denoted by the letters R. F. The student is advised to draw them in their original lengths.

Besides the ordinary solutions of the exercises, there are inserted certain notes wherever it is though expedient to do so. The chief object of these notes is to make the student understand the subject well.

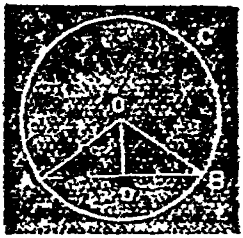
In order to give a clue as to the way in which the construction was discovered, it has been thought more instructive, in the case of a few problems on the construction of triangles, to adopt the method of *analysis* than the ordinary method of *synthesis*.

RAMPUR :
July 1910. } K. L. S.

PART III.

Exercises on Theorem 31, P. 145.

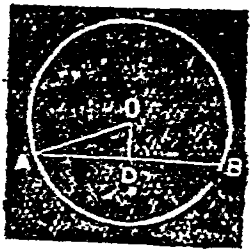
R. F. $\frac{6}{20}$



1. Take a line $AB=8$ cm. Bisect it D, and draw DO perp. to AB making $D=3$ cm. Join OA and OB. With centre and radius OB describe the circle ABC. en *ABC is the required circle.*

$$OB=\sqrt{(OD^2+DB^2)}=\sqrt{(9+16)}=\sqrt{(25)}=5 \text{ cm.}$$

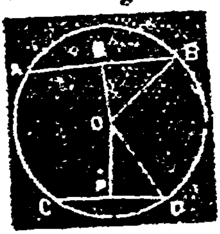
R. F. $\frac{3}{100}$



2. With any pt. O as centre and dius$=13''$ draw a circle. From O draw line $OD=5''$, and through D draw a ord AB perp. to OD. Then AD $=\sqrt{(OA^2-OD^2)}=\sqrt{(169-25)}=\sqrt{(144)}=12''$

$AB=2AD=24''$.

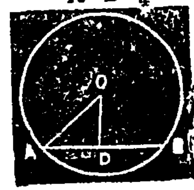
R. F. $\frac{2}{3}$



3. With any pt. O as centre and dius$=1''$ draw a circle, and place in it two hords AB, CD measuring $16''$ and $1.2''$ respectively. From O draw OP, OQ perps. to D and AB respectively. Join OB and OD. hen

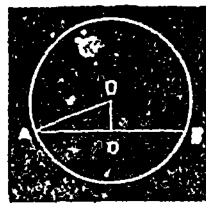
$Q=\sqrt{(OB^2-BQ^2)}=\sqrt{(1-.64)}=\sqrt{(.36)}=.6''$, and $P=\sqrt{(OD^2-PD^2)}=\sqrt{(1-.36)}=\sqrt{(.64)}=8''$

R F $\frac{1}{4}$



4. With any pt. O as centre and radius = 4cm describe a circle and place in it a chord AB=6 cm. From O draw OD perp. to AB Then $OD = \sqrt{(OA^2 - AD^2)} = \sqrt{(16-9)} = \sqrt{(7)} = 2\ 6$cm

R. F. $\frac{3}{10}$



5 Draw the figure as in Ex 4, making the radius=3.7 cm and place in it a chord AB=7 cm Then

$OD = \sqrt{(OA^2 - AD^2)} = \sqrt{(13.69 - 12\ 25)} = \sqrt{(1\ 44)}$
$= 1\ 2$ cm

∴ The true distance=12″ or 1 ft.

R. F $\frac{3}{10}$

6. Draw the figure as in Ex 4, making the radius=1 3″, and place in it a chord AB =2.4″ Then

$OD = \sqrt{(OA^2 - AD^2)} = \sqrt{(1.69 - 1.44)} = \sqrt{(.25)}$
$= 5''$

∴ Area of the △ $OAB = \frac{1}{2}\ AB\ OD$

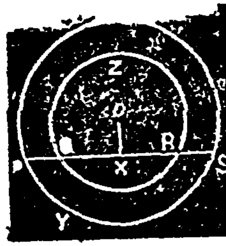
$= \frac{1}{2} \times 2\ 4 \times 5 = 6$ sq in.

R F $\frac{1}{4}$

7 Let P, Q be two given points 3″ apart. Join PQ, and bisect it in R and draw RX perp. to PQ. With centre P and radius 1 7″ draw an arc cutting RX in O Join OP. Then with centre O and radius OP describe a circle PQS. Then

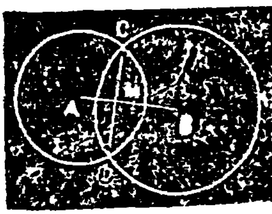
$$OR = \sqrt{(OP^2 - PR^2)} = \sqrt{(2.89 - 2.25)} = \sqrt{(.64)} = .8''.$$

Exercises on Theorems 31-32, P. 147.



1. Let *PYS* and *QZR* be to concentric circles and *O* their common centre. Also let *PQRS* be a straight line cutting the two circles as shown in the diagram. *Then PQ shall be equal to RS.*

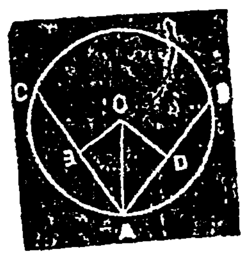
From *O* draw *OX* perp. to *PS*. Then $QX = RX$ and $PX = SX$ [*Th. 31*] $\therefore$ $PX - QX = SX - RX$ i e. $PQ = RS$.



2. (i) Let two circles whose centres are *A* and *B* intersect at *C* and *D*. Join *CD* and bisect it at *M*. *Then AM, BM, shall be in the same straight line.*

Join *AM* and *BM*. Then because the st. line *AM* passing through the centre *A* bisects the chord *CD*, therefore the $\angle AMC$ is a right angle; [*Th. 31*] For the same reason the $\angle BMC$ is a rt angle. Therefore the $\angle$s *AMC BMC* together$=$2 rt. angles, and therefore *AM* and *BM* are in the same st line [*Th. 2*].

(ii) Because the st line *AB* is perp. to *CD* at its middle pt. *M* [*proved above*] $\therefore$ The st. line joining the centres bisects the common chord at right angles.

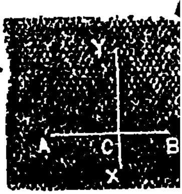


3. Let *AB*, *AC* be any two equal chords of a circle *ABC* whose centre is *O*. *It is required to show that the bisector of the $\angle BAC$ passes through O.*

From *O* draw *OD*, *OE* perpendiculars to *AB* and *AC* respectively, then *AB* and *AC* are bisected at *D* and *E* [*Th 31*] Since $AB = AC$

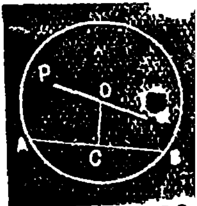
∴ Their halves AD and AE are also equal.

Join OA, then in the right-angled $\triangle s$ AOD and AOE because $AD = AE$ and AO is common to both ∴ the triangles are identically equal [*Th 18*], and therefore the $\angle DAO =$ the $\angle EAO$, hence OA bisects the $\angle BAC$ ¶ ∴ The bisector of the $\angle BAC$ passes through the centre O.



4 Let A and B be any two given points *It is required to find the locus of the centres of all the circles passing through A and B*

Join AB and bisect it at C. From C draw YCX at right angles to AB Then every point on YCX is equally distant from A and B [*Prob. 14*] Also the centre of every circle passing through A and B is equally distant from A and B ¶Hence the locus of the centres of all the circles passing through A and B is the st. line YCX which bisects AB at right angles.

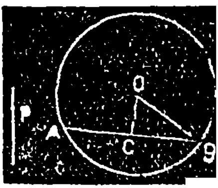


5 Let A and B be any two given points and PQ a given st line *It is required to describe a circle passing through the pts A and B and having its centre on PQ*

Bisect AB at C and draw CO at rt angles to AB Then the centre of the required circle lies on CO. [*Ex 4*]

The centre also lies on the line PQ [*Hyp*]¶ ∴ the point O where CO cuts PQ is the required centre. With centre O and radius OB describe the required circle.

This problem is impossible when the given st line PQ is parl. to the st line CO.



6. Let A and B be any two given points and P a given straight line. *It is required to describe a circle passing through A and B and having its radius* $=P$.

Join AB and bisect it at C. From C draw CO perp. to AB. Then the centre

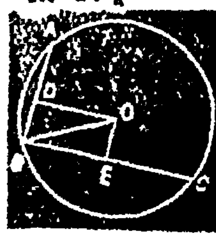
lies on CO. [*Ex.* 4] With centre A and radius $=P$ draw an arc cutting CO at O. Then O is the centre of the required circle. With centre O and radius OA describe the required circle.

This problem is impossible if the given st. line P is less than AC *i. e.* less than half the st. line joining the two given points.

Exercises on theorem 33, P. 149.

1. Let AB, BC be two given st. lines at rt. angles to one another and measuring 1.6" and 3" respectively. *It is required to draw a circle passing through A, B and C, and to find its radius.*

R. F. $\frac{1}{2}$



Centre of the circle passing through the pts. A and B lies on the st. line DO which bisects AB perpendicularly.

Similarly the centre also lies on the st. line EO which bisects BC perpendicularly.

∴ The pt O where these two st. lines intersect is the required centre.

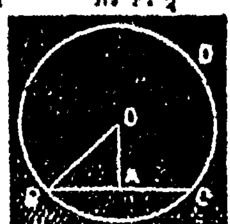
With centre O and radius OA describe a circle then it will pass through B and C.

Radius $OB = \sqrt{(OD^2+BD^2)} = \sqrt{(BE^2+BD^2)} = \sqrt{(2.25+.64)}$

$= \sqrt{(2.89)} = 1.7''$

2. Take any line $OA = 3$ cm. and through A draw BAC perp. to OA and make AB and AC each equal to 3 cm. With centre O and radius OC or OB describe the circle BCD.

R. F. $\frac{1}{2}$



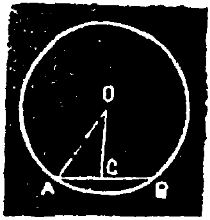
Radius $OB = \sqrt{OA^2+AB^2} = \sqrt{9+9}$

$= \sqrt{(18)} = 4.2$ cm.

3. With any pt O as centre and radius $=4$ cm. describe a circle, and place in it a chord $AB=4$ cm From O draw OC perp to AB. Then OC the distance of the chord from the centre$=\sqrt{(OA^2-AC^2)}$

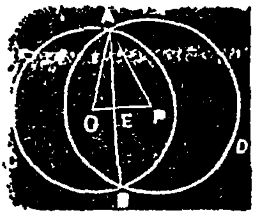
$$=\sqrt{(16-4)}=\sqrt{(12)}=3\ 5 \text{ cm.}$$

R. F. $\frac{1}{4}$



4. Draw a circle ABC whose centre is O and radius $=2\ 5$ cm. Place in it a chord $AB=4\ 8$ cm. From O draw OE perp. to AB then it will bisect it at E [*Th* 31] With centre A and radius $=2.6$ cm draw an arc cutting OE produced at P With centre P and radius $=2\ 6$ cm. draw the circle ABD Join OA and PA. Then

R. F. $\frac{4}{5}$



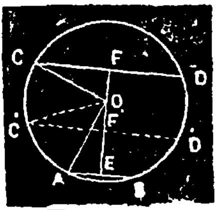
$$OE=\sqrt{(OA^2-AE^2)}=\sqrt{(6\ 25-5\ 76)}=\sqrt{(.49)}=7 \text{ cm.}$$

and $PE=\sqrt{(AP^2-AE^2)}=\sqrt{(6\ 76-5\ 76)}=\sqrt{(1)}=1\ 0$ cm.

$\therefore OP=OE+PE=1\ 7$ cm. Hence the distance between the centres$=17''$

5. With any pt. O as centre and radius $=6\ 5''$ describe a circle and place in it two paralleled chords AB, CD measuring $5''$ and $12''$ respectively From O draw OE perp to BA and produce it to cut CD at F, then OF is also perp. to CD. Join OA and OC

R. F. $\frac{3}{5}$



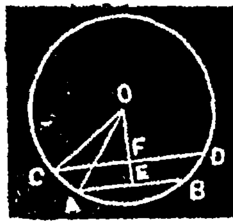
Then, $OE=\sqrt{(OA^2-AE^2)}=\sqrt{(42\ 25-6\ 25)}=\sqrt{(36)}=6,''$ and $OF=\sqrt{(OC^2-CF^2)}=\sqrt{(42\ 25-36)}=\sqrt{(6.25)}=2\ 5''$

$\therefore EF=OE+OF=6''+2.5''=8.5''$

If the chord CD is placed as $C'D'$ on the same side of O as AB, then $EF' = OE - OF' = 6'' - 2.5'' = 3.5.''$

6. Let O be the centre of a circle $ABDC$ and let AB, CD be two parl. chords in it measuring 6 cm. and 8 cm. respectively, and 1 cm. apart from each other. *It is required to find its radius.*

R. F. 1



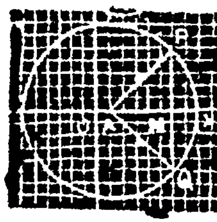
From O draw OE perp. to AB cutting CD at F, and let $OF = x$ cm., then $OE = EF + OF = (1 + X)$ cm.

Join OA, OC. Then $OA^2 = OE^2 + AE^2$, and $OC^2 = OF^2 + CF$

But $OA^2 = OC^2$ $\therefore$ $OF^2 + CF^2 = OE^2 + AE^2$.

i. e. $x^2 + 4^2 = (1 + x)^2 + 3^2$ whence $x = 3$.

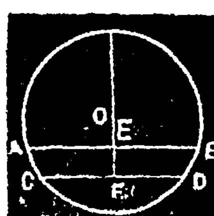
$\therefore$ The radius $OC = \sqrt{(x^2 + 4^2)} = \sqrt{(9 + 16)} = \sqrt{(25)} = 5$ cm.



7. Plot the pts P and Q whose co-ordinates are (6,5) and (6,–5) respectively. Join PQ, and let it cut OX in M, then each of PM and $QM = 5$.

Take any pt. A on OX, and join AP, AQ Then $AP = AQ$. Hence the circle with centre A and passing through P also passes through Q.

8. Let AB and CD be two parallel chords in a circle whose centre is O. From O draw OE perp. to AB, then OE bisects AB.



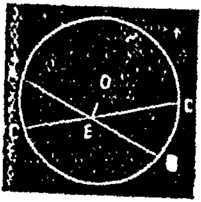
Let OE cut CD at F, then OF is also perp. to CD, and therefore CD is bisected at F.

$\therefore$ The line joining the middle points of two parallel chords passes through the centre.

9. See fig Ex. 8.

If the st. line OEF cuts any other chords parl to AB or CD, it will bisect that chord perpendicularly.

Hence the locus of the middle points of parallel chords is a st. line passing through the centre and bisecting these chords perpendicularly.



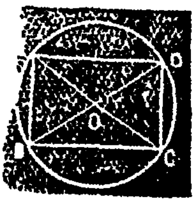
10. If possible let the two intersecting chords AB, CD of the circle whose centre is O, bisect one another at the pt E. Join OE.

Then since $AE=EB$, therefore the $\angle OEB$ is a rt angle [*Th. 31*]

Again since $CE=ED$, therefore the $\angle OED$ is a rt. angle [*Th. 31.*]

$\therefore$ The $\angle OEB$=the $\angle OED$, the less equal to the greate which is impossible.

Hence AB, CD do not bisect each other.



11. Let $ABCD$ be a parallelogram inscribed in a circle, then since the diagonals AC, BD bisect one another at O, hence each of them is a diameter (Ex 10.)

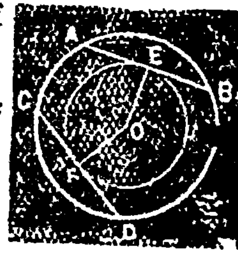
$\therefore$ Their point of intersection O is the centre of the circle.

12. See Fig Ex 11.

Because AC, BD are each of them a diameter, therefore $AC=BD$.

$\therefore$ The parm $ABCD$ is a rectangle [*Ex. 5, p. 58*].

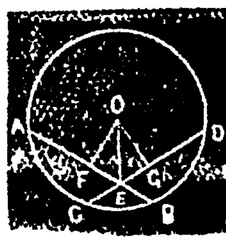
Exercises on Theorem 34, P. 151.



1. Let AB, CD be two of a system of equal chords of a circle whose centre is O. *It is required to find the locus of the middle points of these chords.*

Bisect AB, CD at the points E, F. Join OE, OF. Then $OE = OF$ [*Th. 34*].

∴ The required locus is a circle whose centre is O and radius = the common distance of the equal chords from the centre O



2. Let AB, CD two chords of a circle whose centre is O, intersect at the point E such that the $\angle OEA$ = the $\angle OED$. *Then AB shall be equal to CD.*

Draw OF, OG perps. to AB and CD respectively. Then in the rt. angled $\triangle$s OEF and OEG, because the $\angle OEF$ = the $\angle OEG$, the rt. $\angle OFE$ = the rt $\angle OGE$, and the side OE is common to both.

∴ The triangles are identically equal [*Th. 17*]

∴ $OF = OG$, and therefore $AB = CD$ [*Th. 34*].

3. See figure Ex 2.

If AB is equal to CD, *then AE shall be equal to ED and CE equal to EB.*

Because $AB = CD$, therefore $OF = OG$ (*Th 34*).

Now in the rt. angled triangles OEF and OEG, because $OF = OG$, OE is common to both, and the rt. $\angle OFE$ = the rt. $\angle OGE$.

∴ The triangles are identically equal [*Th. 18*].

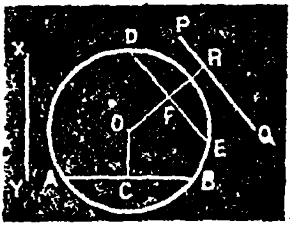
∴ $FE = GE$.

Also $AF=GD$ (*Because they are halves of AB and CD respectively*)

$\therefore AF+FE=DG+GE$, or $AE=DE$.

And $AB-AE=CD-DE$, or $BE=CE$.

4. Let O be the centre of the given circle, PQ and XY two given straight lines, of which XY is not greater than the diameter of the circle. *It is required to draw a chord in the given circle which shall be equal to XY and parallel to PQ.*

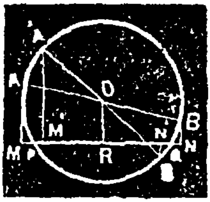


Place a chord $AB=XY$, and from O draw OC perp to AB, and OR perp to PQ. From OR cut off $OF=OC$, and through F draw the chord DFE perp to OR. Then DFE is parl. to PQ.

Again because $OF=OC$, therefore $DE=AB=XY$.

$\therefore DE$ is the required chord.

5. Let AB be the diameter of the given circle, and PQ a fixed chord in it Draw AM, BN perps. to PQ. *Then the sum or difference of AM and BN shall be constant.*



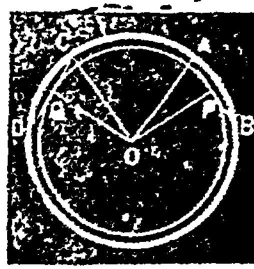
Bisect AB at O, and draw OR perp. to PQ. Then $OR=\frac{1}{2}(AM+BN)$ or $\frac{1}{2}$ $(A'M'-B'N')$ according as A, B are on the same side of PQ or on opposite sides of it [*Ex 9, p 65*].

Since the chord PQ is fixed, therefore its distance OR from the centre is of constant length

Hence the sum or difference of AM and BN is also constant.

6. With any point O as centre, and radius=4.1 cm draw a circle. Place in it two chords AB, CD each=1.8 cm. [*as in Ex. 3 p. 145*].

R. F. $\frac{1}{4}$



Draw OP, OQ perps. to these chords. Then P Q are the middle points of AB and and CD respectively [*Th. 31*].

Because $AB=CD$, therefore $OP=OQ$.

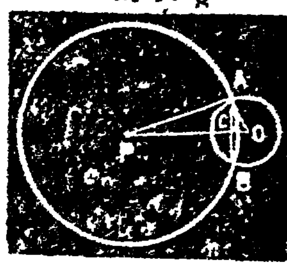
$\therefore$ The points P and Q, *as well as the middle points of all chords 1.8 cm* long, lie on a circle whose centre is O and radius=OP or OQ.

By *Calculation* $OP=\sqrt{OA^2-AP^2}=\sqrt{(4.1)^2-(.9)^2}$ $=4$ cm. *Measure* OP, and you will find it=4 cm. long.

With centre O and radius=4 cm. draw a circle, and notice that it passes through P and Q.

7. Take any two points O and $P=4''$ apart. With centre P and radius=3.7" draw a circle and place in it a chord AB perp. to OP and equal to 2.4" [*as in Ex. 4*]. Then A and B are the points of intersection of this circle with a circle whose centre is O and the common chord=2.4."

R. F. $\frac{1}{8}$



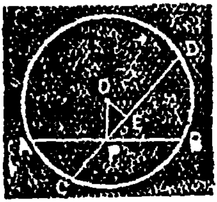
With centre O and radius OA draw this circle, and notice that it passes through B.

Let OP cut AB at C. Then $CP=\sqrt{AP^2-AC^2}$ $=\sqrt{(3.7)^2-(1.2)^2}=3.5.''$

$\therefore OC=OP-CP=4''-3.5''=.5''$.

$\therefore OA=\sqrt{OC^2+AC^2}=\sqrt{(.5)^2+(1.2)^2}=1.3''$.

Exercises on Theorem 35, P. 153.



1. Let O be the centre of the given circle, and let P be a given point in it *It is required to draw the least possible chord through P.*

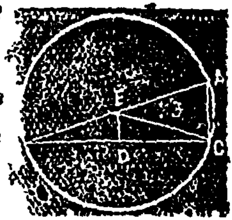
Join OP, and draw the chord APB perp. to OP Then AB shall be the least possible chord

Let CPD be any other chord through P. Draw OE perp to CD Then OP, being the hypotenuse of the rt. angled triangle OEP, is greater than OE

.. CD is greater than AB [*Th 35*]

Similarly it may be proved that every other chord through P is greater than AB.

R F $\frac{1}{4}$

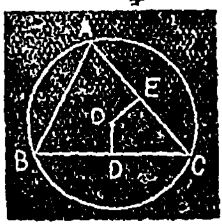


2. Draw the triangle ABC such that AB $=3.7''$, $BC=3.5''$, and $AC=1.2''$ [*Prob 8.*] *It is required to draw a circle having its centre on AB, and passing througth the points B and C.*

Bisect BC at D and draw DE perp to BC meeting AB at E. Join CE With centre E, and radius EC draw the required circle.

Since AB is the diameter of this circle, hence its radius$=\frac{1}{2}$ $AB=1.85''$ ¶ *Measure CE*, and you will find it$=1\ 85.''$

R F $\frac{1}{4}$



3. Draw a triangle ABC such that AB $=2\ 6''$, $BC=3''$, $CA=2\ 8''$ [*Prob. 8*]. *It is required to draw the circum-circle of the triangle ABC.*

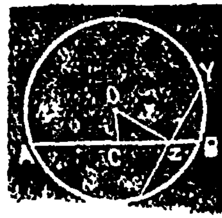
Bisect BC, AC at the points D and E respectively,

Draw DO, EO perps. to BC, AC respectively meeting at the point O. Then O is the centre of the required circle [*Th. 32*].

With centre O, and radius OA describe a circle and notice that it also passes through the points B and C.

Measure OA and you will find it$=1.62''$

4. Let O be the centre of the given circle of which AB is a fixed chord. Let XY be any other chord of this circle having its middle point Z on AB. *It is required to find the greatest and the least length that XY can have, as Z approach is the middle point of AB.*



(i) Join OZ, and draw OC perp to AB, then AB is bisected at C. From the rt angled triangle OCZ, we have the hypotenuse OZ greater than OC. ¶ $\therefore$ XY is less than AB [*Th. 35*].

Similarly it may be shown that any other chord having its middle point on AB is less than AB.

$\therefore$ *AB is the greatest length that XY can have.*

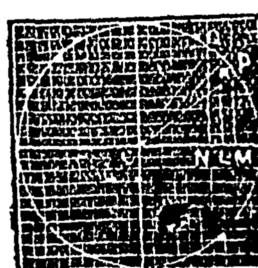
(ii) OZ decreases as Z approaches C [*Th. 12, Cor. 3*].

$\therefore$ XY increases as Z approaches the middle point of AB.

(iii) The greatest length that OZ can have = radius of the circle. But in this case XY becomes tangent to the circle at Z and hence the tangent A or B denotes the least possible length of XY.

5. *Each unit of the fig* $=3''$

With the origin O as centre and radius$=3''$ draw a circle. *It is required to show that this circle passes through the pts. P and Q whose co-ordinates are* $(1.1'', 1.8'')$ and $(1.8'', 2.4'')$.



Draw PM, QN and RL perps. to the X-axis.

(i) Because $OP=\sqrt{OM^2+PM^2}=\sqrt{(2.4)^2+(1.8)^2}=3''$, and $OQ=\sqrt{ON^2+QN^2}=\sqrt{(1.8)^2+(2.4)^2}=3''$

∴ The pts. P and Q lie on this circle

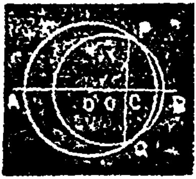
(ii) $PQ=\sqrt{\left\{(2.4-1.8)^2+(1.8-2.4)^2\right\}}=.85''$

(iii) Bisect PQ at R. Then the co-ordinates of R are

$\left(\frac{2.4+1.8}{2},\frac{1.8+2.4}{2}\right)$ *i. e*, $(2.1'',2.1'')$.

(iv) Join OR, then OR represents the distance of PQ from O, hence $OR=\sqrt{OL^2+RL^2}=\sqrt{2.1^2+2.1^2}=2.91''$.

Exercises on Theorem 36. P. 155.

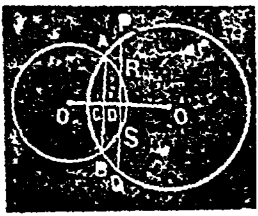


1. Let AB be a given st line, and P a given point. *It is required to show that all circles passing through P also passes through a second fixed point.*

Draw PC perp. to AB, and produce PC to Q making $CQ=PC$. Then Q is a fixed point, and AB bisects PQ perpendicularly.

∴ All the points on AB are equidistant from P and Q [*Prob 14*].

∴ Circles whose centres $O.O'$ etc. lie on AB and which pass through P, also pass through Q



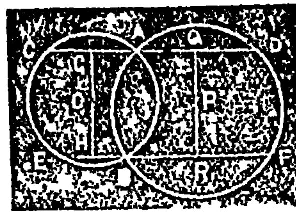
2. Let AB be the common chord of two circles whose centres are O and O' and let a st. line PQ parl. to AB cut these circles at the pts. P, Q, and R, S Then PR shall be equal to QS.

Let O,O' intersect PQ at D, then since OO' is perp. to AB, it is also perp. to PQ. Therefore the chords PQ and RS are both bisected by OO', [*Th*, 31]

∴ $PD = QD$ and $RD = SD$.

∴ $PD - RD = QD - SD$, that is $PR = QS$.

3 Let any two circles whose centres are O, P intersect at the pts. A and B, and let there be drawn two parl. st lines GAD, EBF to cut the circles *Then CD shall be equal to EF.*



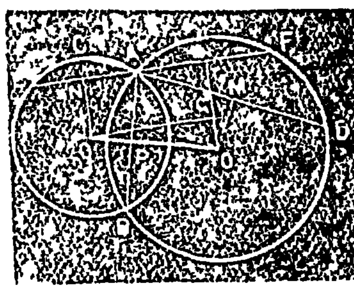
From O and P draw OG, PQ perps. to CD, and produce them to meet EF at the points H and R respectively Then $GHRQ$ is a rectangle, and therefore $GQ = HR$ (*Th. 21*).

Since $AG = \frac{1}{2}\,AC$, and $AQ = \frac{1}{2}\,AD$, hence $GQ = \frac{1}{2}\,CD$.

Similarly $HR = \frac{1}{2}\,EF$.

But $QG = HR$ therefore $CD = EF$.

4. Let O, O' be the centres of two circles cutting at the points A and B, and through A let there be drawn two st. lines CAD, EAF making equal angles with AB and terminated by the circumferences. *Then CD shall be equal to EF.*



Join OO' cutting AB at P. Then AB is bisected perpendicularly by OO' at P.

Draw OM; $O'M'$ perps. to CD, and ON, $O'N'$ perps. to EF. also draw $O'G'$ parl. to CD meeting OM in G', and draw OG parl to EF meeting $O'N'$ at G.

In the quadrilateral $AMOP$, because the $\angle$s at M and P are rt. angles, therefore the $\angle$s MAP and MOP are supplementary.

Similarly it may be proved that the $\angle$s $N'AP$, $N'O'P$ are supplementary

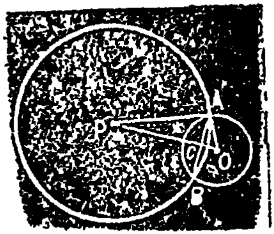
But the $\angle MAP =$ the $\angle N'AP$, therefore the $\angle MOP =$ the $\angle N'O'P$.

Now in the rt. angled triangles $OO'G$ and $OO'G'$, because

the $\angle OO'G$ = the $\angle O'OG'$, the rt. $\angle OGO'$ = the rt $\angle O'G'O$ and the side OO' is common to both, hence the triangles are identically equal [*Th 17*] ∴ $OG=O'G'$

But $OG=NN'$ and $O'G=MM'$, ∴ $MN'=NN'$. Again $NN'=\frac{1}{2}$ EF, and $MM'=\frac{1}{2}$ CD (*Ex 4*) ∴ $CD=EF$.

5. Take a line $AB=2.4$ cm Bisect it at C and draw a line OP perp to AB at the point C From the centre A and radii equal to 2" and 3 7" cut the line OP at the points O and P respectively. With O and P as centres and radii=2" and 3 7" draw two circles *It is required to find the length of OP*



Join OA, PA Then $OC=\sqrt{(OA^2-AC^2)}=\sqrt{4-1.44}=1.6$ cm, and $PC=\sqrt{(PA^2-AC^2)}=\sqrt{\{(3.7)^2-(1.2)^2\}}=3.5$ cm.

∴ $OP=OC+PC=1.6+3.5=5.1$ cm

∴ The distance between the centres=5.1".

6 See Fig Ex 5

Take any two points O and P 2.1" apart. With centres O and P and radii=1" and 1.7" respectively draw two circles cutting one another at the points A and B Join OP, and let it intersect AB at C *It is required to find AB, OC, and PC.*

Let $PC=x$, then $OC=(OP-PC)=(2.1-x)$

Then because $OA^2-OC^2=AC^2=AP^2-PC^2$

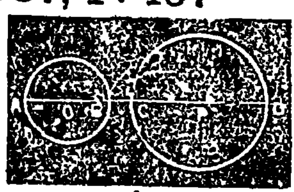
∴ $1^2-(2.1-x)^2=(1.7)^2-x^2$, whence $x=1.5"$

∴ $OC=(2.1-x)=(2.1-1.5)=.6"$

∴ $AC=\sqrt{(AO^2-OC^2)}=\sqrt{(1-.36)}=.8"$, and therefore $AB=2AC=1.6"$

Exercises on Theorem 37, P. 157

1 Let O, P be the centres of two given circles which do not intersect. Join OP cutting the circles at B and C and produce OP both ways to cut the circles again at A and D. *Then*

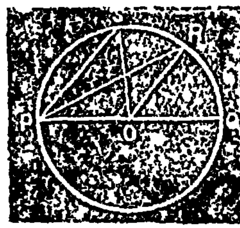


AD shall be the greatest and BC the least of the straight lines which have one extremity on each of the given circles

The greatest st line which has one extremity on each of the two given circles must be that which passes through the two centres [*Th 37*]. Therefore *AOPD* is the greatest of such st. lines

The least st line which has one extremity on each of the two given circles must be that which when produced passes through the two centres [*Th 37*]. Therefore *BC* is the least of such st lines

2 Let *P* be any point on the circumference of a circle whose centre is *O*, and from *P* let there be drawn a diameter *PQ*, and two other chords *PR*, *PS* of which *PR* subtends a greater angle at *O* than *PS*. *Then PQ shall be the greatest chord and the chord PR shall be greater than PS*



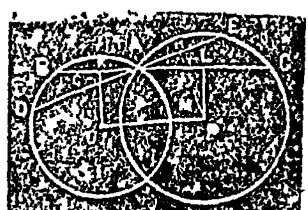
Join *OR* *OS*. Then because *PO*, *OR* are together greater than *PR* [*Th 11*]. And *OR* = *OQ* therefore *PQ* is greater than *PR*

Similarly it may be proved that *PQ* is greater than *PS* or any other chord drawn from *P*.

∴ *PQ* is the greater chord.

In the △s *POR* and *POS*, because the two sides *PO*, *OR* are equal to the two sides *PO* and *OS*, each to each, but the ∠*POR* is greater than the ∠*POS* Therefore *PR* is greater than *PS*. [*Th 19*].

3. Let *A* be a point of intersection of two circles whose centres are *O* and *P*. Join *OP*, and through *A* draw a st. line *BAC* parl. to *OP* and terminated by the circumferences. *Then BC shall be the greatest of all such straight lines drawn through A.*



Let DAE be any other such st line drawn through A. From O and P draw OK, OG and PL, PH perpendiculars to BC and DE respectively Also from O draw OM parl. to DE meeting PH in M.

Because the hypotenuse OP is greater than OM, and $OM=GH$ and $OP=KL$ [*Th 21*]. Therefore KL is greater than GH

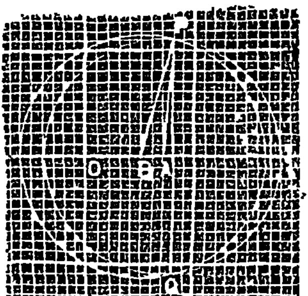
But $KL=AK+AL=\frac{1}{2}AB+\frac{1}{2}AC=\frac{1}{2}BC$.

and $GH=AG+AH=\frac{1}{2}AD+\frac{1}{2}AE=\frac{1}{2}DE$

$\therefore$ BC is greater than DE.

Similarly it may be proved that BC is greater than any other st line drawn through A and terminated by the circumferences

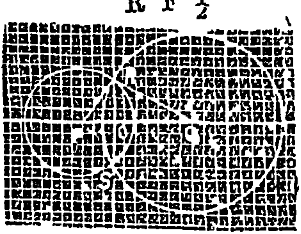
$\therefore$ BC is the greatest of all such straight lines.



4 Take any two points A,B on the x axis, and let P be the point whose coordinates are (8, 11) With centres A and B and radii AP and BP respectively draw two circles intersecting again at the point Q. *It is required to find the coordinates of Q*

Join PQ then it is bisected by the x axis [*Th 31*] hence the coordinates of Q are (8 –11)

R F $\frac{1}{2}$



5 Plot the points P, Q and R whose coordinates are (–6, 0), (15, 0) and (0, 8) With centres P and Q and radii PR and QR respectively draw two circles intersecting again at S. *It is required to find the coordinates of S and the lengths of their radii.*

Because the centres P and Q both lie on the x-axis, therefore the st. line RS is bisected perpendicularly by the x axis, and therefore the coordinates of S are (0–8),
$PR = \sqrt{(OP^2+OR^2)} = \sqrt{(36+64)} = 10$,
and $QR = \sqrt{(OQ^2+OR^2)} = \sqrt{(225+64)} = 17$.

6. Let OAB be an isosceles triangle having the vertical angle $O=80°$. With centre O and radius OA draw a circle, and let P, Q, R .. be any number of points on the circumference of this circle and on the same side of AB as the centre O. Join AP, BP, AQ, BQ; AR, BR ..

Measure the angles APB, AQB, ARB... and you will find each of them to be equal to 40°.

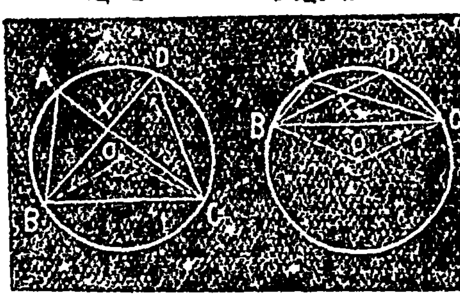
Now take $\angle AOB=60°$, and do the same construction as given above You will then find that each of the angles APB, AQB, $ARB=30°$.

∴ *The angles subtended by any chord at the circumference of a circle are all equal to one another, and each of them is equal to half of the angle subtended by the chord at the centre.*

Exercises on Theorem 39, P. 161.

1. If the $\angle BDC=74°$, then the $\angle BAC=$ the $\angle BDC=74°$, and the $\angle BOC$ $=2$ the $\angle BDC$, $=148°$ Therefore the $\angle OBC$, being half the supplement of the $\angle BOC = \frac{1}{2}$ (180°–148°)=16°.

Fig 1 Fig. 2



2. See Ex. 1 Fig. 2

Because the $\angle s$ XDC, XCD, DXC together $=180°$ [*Th. 16*] Therefore the $\angle XDC=180°$–the two $\angle s$ XCD and $DXC=180°$ –(40°+25°)=115°.

∴ The $\angle BAC$=the $\angle BDC$=115°, and the reflex $\angle BOC$ =2 the $\angle BDC$=230°.

3. See Fig 1. Ex. 1.

Because the $\angle s$ CBD, BCD, BDC together=180° [*Th 16*] Therefore the $\angle BDC$=180°–the two $\angle s$ CBD and BCD=180° –(43°+82°)=55°.

∴ The $\angle BAC$=the $\angle BDC$=55°, and the $\angle BOC$=2 the $\angle BDC$=110°

Each of the $\angle s$ OBC and OCB, being half the supplement of the $\angle BOC$=½ (180°–110°)=35°.

∴ The OBD=the $\angle CBD$–the $\angle OBC$=43°–35°=8°, and the $\angle OCD$=the $\angle BCD$—the $\angle OCB$=82°—35°=47°

4 See Fig. 2 Ex. 1

It is required to prove that $\angle BOC = \angle BAC - 90°$

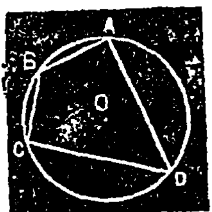
Because $\angle OBC + \angle OCB + \angle BOC = 180°$ and $\angle OBC = \angle OCB$, therefore $2\angle OBC = 180° - \angle BOC$, and because $\angle BOC$ $= 360°$–ref $\angle BOC$

∴ $2\angle BOC = 180° - (360° - \text{ref } \angle BOC) = \text{ref } \angle BOC - 180°$

$\angle BOC = \frac{1}{2}$ ref $\angle BOC - 90° = \angle BAC - 90°$.

Exercises on Theorem 40. P 163.

1 With any point O as centre and radius =1 6″ draw a circle Take any point B on the circumference, and make the $\angle ABC$ =126°, the arms of which cut the circumference at the points A and C



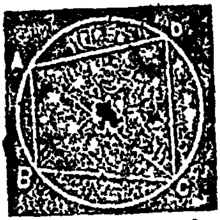
Take any other point D on the circumference, and join AD, CD *Then ABCD is the required quadrilateral*

Measure the $\angle ADC$, and you find it=54°

Measure also the $\angle s$ BAD, BCD and you will find them =74° and 106° respectively

∴ The opposite angles BAD, BCD of a cyclic quadrilateral are supplementary.

2 Let $ABCD$ be any cyclic quadrilateral. *It is required to prove that the* $\angle$s ADC, ABC *together* $=2$ *rt angles, as well as the* $\angle$s BAD, BCD *together* $=2$ *rt. angles.*

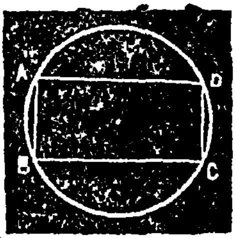


Join AC and BD. Because the $\angle BDC=$the $\angle BAC$, and the $\angle ACB=$the $\angle ADB$ [*Th.* 39].

∴ The $\angle ADC=$the $\angle ADB+$the $\angle BDC=$the $\angle ACB+$the $\angle BAC$. But the $\angle$s ACB, BAC and ABC together$=2$ rt. angles [*Th* 16]. Therefore also the $\angle$s ADC and ABC together $=2$ rt. angles.

Similarly it may be proved that the $\angle$s BAD and BCD together $=2$ rt angles.

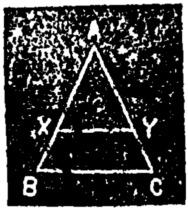
3. Let $ABCD$ be a parallelogram about which a circle can be described. *Then* $ABCD$ *shall be a rectangle.*



Because $ABCD$ is a cyclic quadrilateral, therefore the $\angle$s BAD, BCD together$=2$ rt angles. [*Th* 40] Again because $ABCD$ is a parm. Therefore the $\angle BAD=$ the $\angle BCD$.

∴ Each of the $\angle$s BAD, BCD is a rt. angle. ∴ $ABCD$ is a rectangle.

4. Let ABC be an isosceles triangle, and let there be drawn a st. line XY parallel to BC cutting AB, AC at the points X and Y respectively. *Then the four points* B, C, X, Y *shall lie on a circle.*

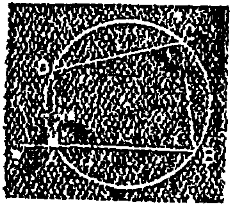


Because the $\angle$s YXB, XBC together$=2$ rt. angles [*Th* 14], and the $\angle XBC=$the $\angle YCB$ [*Th*, 5].

∴ The ∠s YXB and YCB together=2 rt angles

Similarly it may be proved that the ∠s XYC and XBC together = 2 rt. angles.

∴ The points B, C, X and Y lie on a circle [*Converse of Th* 40].



5. Let $ABCD$ be a cyclic quadrilateral of which the side BA is produced to E. *Then the exterior angle EAD shall be equal to the interior and opposite angle BCD.*

Because $ABCD$ is a cyclic quadrilateral,

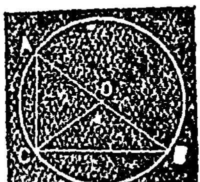
therefore the ∠s BAD and BCD together=2rt. angles [*Th* 40]

Also the ∠s BAD and EAD together=2 rt angles [*Th* 1].

Therefore the ∠s BAD, BCD together=the ∠s BAD, EAD

Take away the common ∠BAD from each of these equals, then the ∠BCD=the ∠EAD

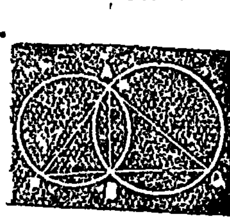
Exercises on Theorem 41, P 165



1 Let ACB be a triangle right-angled at C. *It is required to prove that the circle described on the diameter AB passes through C*

Bisect AB at O, and join OC. Then OC=½ AB [*Ex* 10, *p* 47] ∴ OA=OB=OC.

∴ A circle described with centre O and radius OA will pass through the points B and C [*Th* 33].

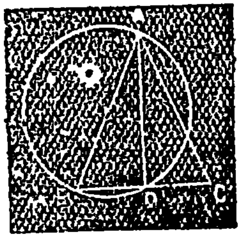


2 Let A and B be the points of intersection of any two circles, and let there be drawn two diameters AP, AQ one in each circle. *Then the points P, B and Q shall be collinear*

Join AB, PB and QB. Then because AP is a diameter therefore the $\angle ABP$ is a rt angle [*Th* 41]

Again because AQ is a diameter, therefore the $\angle ABQ$ is a rt. angle.

$\therefore$ The $\angle s$ ABP and ABQ together $=2$ rt. angles. $\therefore$ PBQ is one straight line [*Th* 2].



3. Let ABC be an isosceles triangle, and on one of the equal sides AB as diameter a circle is described cutting BC at the pt D. *Then D shall be the middle point of BC.*

Join AD. Then because AB is a diameter therefore the $\angle ADB$ is a right angle [*Th* 41], and therefore AD is perp to BC

In the right-angled $\triangle s$ ABD and ACD because the hypotenuse $AB =$ the hypotenuse AC, and the side AD is common to both therefore the triangles are identically equal (*Th* 18).

$\therefore$ $BD = CD$ and therefore D is the middle point of BC.

4 See Figure Ex. 2

Let APQ be a triangle, and let AB be the perpendicular drawn from A to PQ *It is required to prove that the circles described on AP, AQ as diameters will intersect at B.*

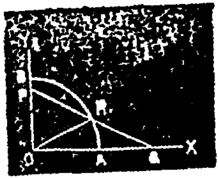
Because ABP is a rt angled triangle and AP is its hypotenuse, therefore the circle described on the diameter AP will pass through B [*Ex.* 1]

Similarly it may be proved that the circle described on the diameter AQ will pass through B.

$\therefore$ The circles described on two sides of a triangle as diameters intersect on the third side at a point where the perp. from the opposite angular point cuts the third side.

The pt. B will be on PQ produced if the $\triangle APQ$ is obtuse angled at P or Q

5. Let PQ represent one position of the straight rod sliding between two straight rulers OX and OY fixed at right angles to one another.



It is required to find the locus of the middle point of the rod PQ.

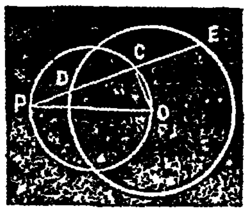
Bisect PQ at R, and join OR Then $OR=\frac{1}{2}PQ$, (*Ex. 10,P.47*)

Because PQ is of constant length therefore OR is also constant Also O is a fixed point.

$\therefore$ The locus of R is a circle whose centre is O and radius $OR=\frac{1}{2}PQ$

But since the rod PQ always slides between the rulers, therefore its middle point R never goes beyond these rulers. Hence the required locus is a quadrant ARB of this circle lying between OX and OY

6. Let O be the centre of the given circle and P a given point *It is required to find the middle point of the chords of the given circle drawn through P*



Through P draw any chord PDE and bisect DE at C Join OC Then OC is perp to DE (*Th. 31*)

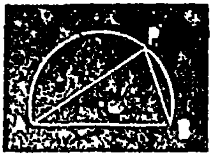
$\therefore$ PCO is a right-angled triangle, and hence a circle described upon the diameter OP passes through C (*EX 1*)

The same is true for the middle point of any other chord drawn through P.

The locus of the middle points of all chords drawn through P is a circle described on the diameter OP

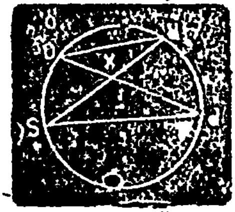
OP is *less than, equal to, or greater* than the radius of the given circle according as the pt P lies *within, on* or *without the circumference.*

Exercises on angles in a circle, P 170-71



1 Let P be any point on the arc of a segment of which AB is the chord *Then the sum of the $\angle$s PAB and PBA shall be constant.*

Sum of the $\angle$s PAB, PBA and $APB=180°$ [*Th 16*] therefore the $\angle PAB+$the $\angle PBA=180°-$the $\angle APB$. But the $\angle APB$ is the constant ¶ $\therefore$ The sum of the $\angle$s PAB and PBA is also constant

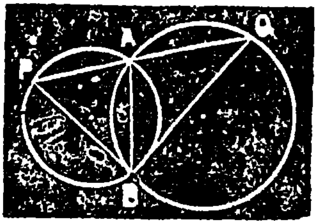


2 Let PQ and RS be any two chords of a circle intersecting at X. *Then the triangles PXS and RXQ shall be equiangular to one another*

Because the $\angle PSR=$the $\angle PQR$, and the $\angle SPQ=$the $\angle SRQ$ [*Th 39*]

Also the $\angle PXS=$the $\angle RXQ$

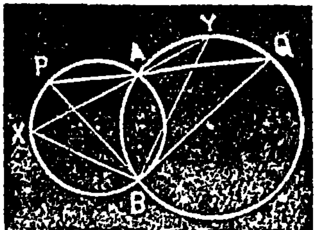
$\therefore$ $\triangle$s PXS and RXQ are equiangular to one another.



3 Let there be any two circles intersecting at A and B, and let through A there be drawn any st line PAQ terminated by the circumferences *It is required to prove that PQ subtends a constant angle at B*

Since the chord AB is constant, therefore the $\angle$s BPA and BQA are constant

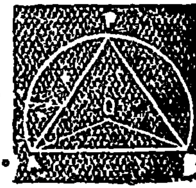
$\therefore$ The $\angle$ PBQ being the supplement of the sum of $\angle$s BPA and BQA [*Th 16*] is also cons'



4 Let there be any two circle intersecting at A and B and through A let there be drawn any two st. lines PAQ, XAY terminated by the circumferences. *It is required to prove that the arcs PX and QY subtend epual angles at B.*

Join PB, XB, QB and YB. Then the $\angle PBQ$=the $\angle XBY$ [*Ex.* 3]. From each of them take away the common $\angle PBY$ ¶ ∴ The remaining $\angle$s PBX and QBY are also equal

5. Let there be any point P on the arc of a segment whose chord is AB and let the bisectors of the $\angle$s PAB and PBA intersect at O *It is required to find the locus of O.*



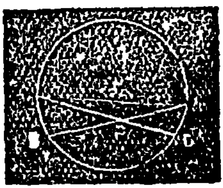
From the $\triangle OAB$, the angles $O+\frac{1}{2}A+\frac{1}{2}B=180°$ ... .(i),¶ and from the $\triangle APB$ the angles $P+A+B=180°$ and therefore $\frac{1}{2}P+\frac{1}{2}A+\frac{1}{2}B=90°$. (ii)

Subtracting (ii) from (i) we have $O-\frac{1}{2}P=90°$.

∴ $O=90°+\frac{1}{2}P$. But the $\angle P$ is constant, therefore the $\angle O$ is also constant.

∴ The locus of O is an arc of a segment on the fixed chrod AB, and containing an angle $=90°+\frac{1}{2}P$

6. Let there be any two chords AB, CD intersecting within the circle at the pt P *Then the $\angle BPC$ shall be equal to the angle subtended at the centre by half the sum of the arcs AD and CB*

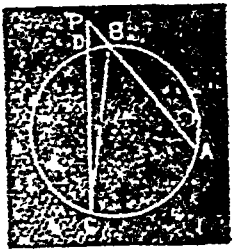


Since the angle subtended at the centre by half an arc= the angle subtended at the circumference by the whole arc

∴ The angle subtended at the centre by half the sum of the arcs AD and BC=the sum of the angles subtended at the circumference by the arcs AD and BC *i.e* =the sum of the $\angle$s ACD and BAC

But the ext $\angle BPC$=the sum of the $\angle$s ACD and BAC.

∴ The $\angle BPC$=the angle subtended at the centre by half the sum of the arcs AD and BC.

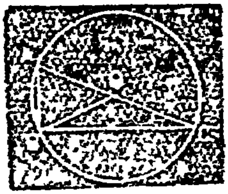


7. Let there be any two chords AB, CD intersecting without the circle at the pt P. *Then the $\angle BPD$ shall be equal to the angle subtended at the centre by half the difference of the arcs AC and BD.*

Since the angle subtended at the centre by half an arc = the angle subtended at the circumference by the whole arc

∴ The angle subtended at the centre by half the difference of the arcs AC and BD=the difference of the angles subtended at the circumference by the arc AC and BD *i. e.* the difference of the $\angle$s ABC and BCD.

But the $\angle BPD$=the ext. $\angle ABC$-the int. and opp. $\angle BCD$ (*Th. 16*), therefore the $\angle BPD$=the angle subtended a tthe centre by half the difference of the arcs AC and BD.

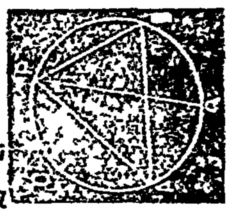


8 Let BA and BC be any chords at right angles to one another. *Then the sum of the arcs cut off by BA and BC shall= to the semi-circumference.*

Since the $\angle ABC$ is a rt. angle.

∴ The segment ABC is a semi-circle.

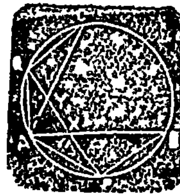
∴ The arcs cut off by AB and C together=semi-circumference.



9. Let AB be any fixed chord of a circle and P any point on the arc ACB cut off by AB. Join PA, PB and let the bisector of the $\angle APB$ meet the conjugate arc at Q. *Then for all positions of P, the pt. Q shall be a fixed point.*

Since the $\angle APQ$=the $\angle BPQ$, therefore the arc AQ=the arc BQ [*Th 42*]

∴ Q being the middle point of the arc AB is a fixed point.



10 Let AB and AC be any two chords of a circle Bisect the minor arcs AB and AC at the pts P and Q, and join PQ cutting AB at X and AC at Y. *Then shall* $AX=AY$.

Since $\angle AQP = \angle ABP$ in the same segment [*Th* 39],

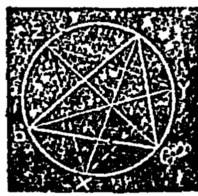
$= \angle PAB$ [$AP=PB$].

And $\angle APQ = \angle ACQ$ in the same segment [*Th* 39],

$= \angle CAQ$ [$\because AQ=QC$]

$\therefore$ The third angles of the $\triangle s$ AQY and APX are also equal, *viz* $\angle AYQ = \angle AXP$

$\therefore$ Their supplements are also equal, *viz* $\angle AYX = \angle AXY$ and therefore $AY=AX$



11 Let ABC be any triangle inscribed in the circle ABC, and let the bisectors of the $\angle s$ A, B and C meet the circumference at the pts X, Y and Z respectively Join XY, YZ, and ZX. *Then the angles of the triangle XYZ shall be equal to* $90^\circ - \frac{a}{2}$, $90^\circ - \frac{b}{2}$ and $90^\circ - \frac{c}{2}$.

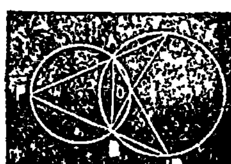
The $\angle ZXY = \angle ZXA +$ the $\angle AXY$

$=$ the $\angle ZCA +$ the $\angle ABY$ [*Th* 39]

$= \frac{c}{2} + \frac{b}{2}$

$= 90^\circ - \frac{a}{2}$ [$\because \frac{a}{2}+\frac{b}{2}+\frac{c}{2} = 90^\circ$ *Th* 16]

Similarly it can be proved that $\angle XZY = 90^\circ - \frac{c}{2}$ and $\angle XYZ = 90^\circ - \frac{b}{2}$

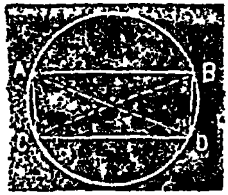


12 Let any two circles APB, AQB intersect each other at A and B, and let P be any point on the circle APB. Join PA and PB and produce them to meet the circle AQB in the pts. Q and R. *Then the arc QR shall be constant.*

Join AB and AR, then since AB is a fixed chord, therefor $\angle s$ APB, ARB are constant.

∴ ext. $\angle QAR$=the $\angle APB$+the $\angle ARB$ [Th 16] is also constant.

∴ The arc QR is of constant length



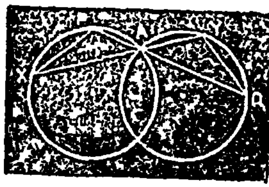
13. Let AB and CD be any two parallel chords of a circle $ACDB$ Join AC, BD, AD and BC. *Then shall $AC=BD$, and $AD=BC$*

Since the $\angle ABC$=the alt $\angle BCD$ (Th.14) therefore the arc AC=the arc BD (Th 42), and therefore the chord AC=the chord BD (Th. 45)

Again because the $\angle$s BAC, BDC are supplementary (Th 40). as also the $\angle$s BAC, ACD are supplementary (Th 14)

∴ The $\angle BDC$=the $\angle ACD$, and therefore the arc BC= the arc AD.

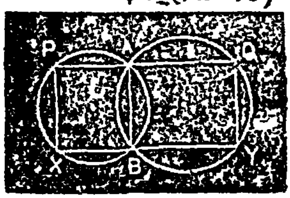
∴ The chord BC=the chord AD (Th 45)



14. Let two equal circles XPA and QYA intersect at the pt. A, and let PAQ and XAY be any two st lines drawn from A and terminated by the circumferences.

Then the chord PX shall be equal to the chord QY.

Because the $\angle PAX$=the $\angle QAY$, therefore the arc PX=the arc QY (Th 42), and therefore the chord PX=the chord QY (Th 45)

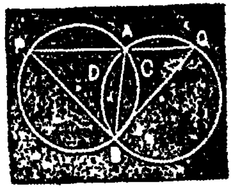


15. Let any two equal circles APX and AQY intersect at the pts. A and B, and let PAQ and XBY be any two parallel st lines drawn through A and B and terminated by the circumferences. *Join PX and QX Then PX shall be equal to QY.*

Join AB, then since PA is parl. to XB, therefore $PX=AB$ (Ex. 13)

Similarly $QY=AB$ (*Ex. 13*) therefore $PX=QY$

16. Let two equal circles *ACB* and *ADB* intersect each other at the pts. *A* and *B* and let *PAQ* be any st line drawn through *A* and terminated by the circumferences

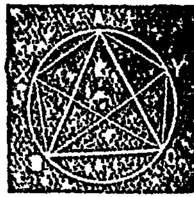


Then BP shall be equal to BQ

Join *AB* Since the circles are equal therefore the chorde *AB* will cut equal arcs *ACB* and *ADB* (*Th* 44)

$\therefore$ The angles standing on these equal arcs are equal, *viz* the $\angle APB$=the $\angle AQB$ (*Th*. 43) Hence $BP=BQ$

17. Let *ABC* be an isosceles triangle inscribed in a circle and let the bisectors of the $\angle$s *C* and *B* meet the circumference at the points *X* and *Y* respectively *Then BX, XA, AY and YC shall be equal to one ananother*



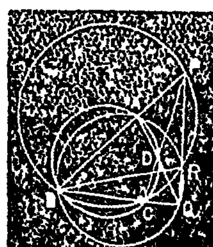
Because the $\angle ABC$ = the $\angle ACB$ and *BY, CX* bisect them, therefore the $\angle$s *ABY CBY, ACX* and *BCX* are all equal.

¶ $\therefore$ The arcs on which they stand are also equal (*Th* 42), and the chords which cut off these equal arcs are equal (*Th* 45) *viz*, $\therefore$ the chords *AY, CY, BX* and *AX* are all equal to one another

(ii) In order that the fig *BXACY* be equilateral, the angles subtended by its sides must be equal, and therefore the $\angle BAC$=the $\angle YBC=\frac{1}{2}B=\frac{1}{2}C$.

Hence the vertical angle must be half of each of the base angles

18. Let *ABCD* be a cyclic quadrilateral, and let *BA,CD* produced meet at *P*, and *BC, AD* at *Q* If the circumcircles of the triangles *PBC, QAB* intersect at *R* *then the points P, R and Q shall be collinear.*

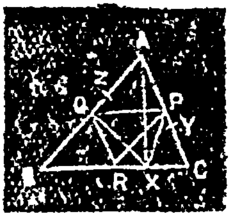


Join *BR*. Then because the $\angle BCP$=

the $\angle$ *BRP* [*Th* 39] and the $\angle$ *BAQ*=the $\angle$ *BRQ* [*Th.* 39] ¶ ∴ The $\angle$ *BRP*+the $\angle$ *BRQ*=the $\angle$ *BCP*+the $\angle$ *BAQ*=180° [*Th* 40].

∴ *PR* and *RQ* are in one st line *i e*, the pts. *P*, *R* and *Q* are collinear.

19. Let *ABC* be a triangle and let *P*, *Q*, *R* the middle pts. of *CA*, *AB* and *BC* respectively. From *A* draw *AX* perp to *BC*. *Then the pts. P, Q, R and X shall be concyclic.*



Join *PQ*, *QR*, *PR*, *PX* and *QX*.

Because *AXC* is a right-angled triangle, and *P* is the middle point of the hypotenuse *AB* therefore *AP*=*PX* [*Prob. 10*] and therefore the $\angle$ *PAX*=the $\angle$ *PXA*

Similarly the $\angle$ *QXA*=the $\angle$ *QAX* therefore the whole $\angle$ *PXQ*=the whole $\angle$ *PAQ*.

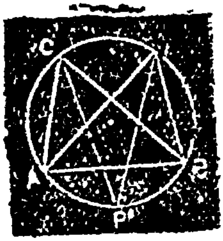
Again because *PAQR* is a parallelogram [*Ex 2 P. 64*], therefore the $\angle$ *PRQ*=the $\angle$ *PAQ*, and therefore $\angle$ *PRQ* = $\angle$ *PXQ*.

∴ The points *P*, *Q*, *R* and *X* are concyclic [*Converse of Th. 39*].

20. See figure Ex. 19.

Let *Y* and *Z* represent the feet of the perpendiculars drawn from *B* and *C* to the opposite sides. Then as in the last exercise it can be proved that the circle passing through the points *P*, *Q* and *R* also passes through the pts *Y* and *Z*. Therefore the middle points of the sides of a triangle and the feet of the perpendiculars let fall from the vertices on the opposite sides are concyclic.

21. Let *AB* be the given base, and *ADB* the given angle, then the vertices of all triangles on this base and having a vertical angle=the $\angle$ *ADB* lie on the arc *ACB* (*Converse of Th. 39*)



∴ The bisectors of the vertical angle shall in all positions

of C pass through P the middle point of the minor arc AB (*Ex. 12*)

22 Let ABC be a triangle inscribed in a circle, and let E be the middle point of the arc subtended by BC on the side remote from A. Draw ED the diameter of the circle, and join AE. *Then the* $\angle AED$ *shall* $=\frac{1}{2}(B-C)$

Join BE, EC, BD and CD Then because the arc $BE=$the arc CE, therefore the $\angle BDE=\angle CDE$, and because ED is the diameter, therefore the $\angle$s DBE, DCE are rt angles

∴ The three angles of the rt angled triangles BED and CED are also equal viz the $\angle CED=$the $\angle BED$.

i.e., the $\angle CEA-$the $\angle AED=$the $\angle BEA+$the $\angle AED$

∴ 2 the $\angle AED=$the $\angle CEA-$the $\angle BEA$.

$=$the $\angle CBA-$the $\angle BCA$.

∴ the $\angle AED=\frac{1}{2}(\angle CBA-\angle BCA)=\frac{1}{2}(B-C)$

Exercises on the Tangent, P. 177.

1. With any point O as centre and radii $=$ 5 cm and 3 cm draw two concentric circles. Draw AB, CD, EF a series of chords of the greater circle which touch the smaller circle at the points P, Q, and R respectively. Join OP, OQ and OR then, these are perpendiculars to AB, CD and EF respectively.

E. I. $\frac{1}{5}$

Because OP, OQ and OR are equal to one another therefore AB, CD and DF are also equal (*Th. 31*).

Join OA, then $AP=\sqrt{(OA^2-OP^2)}=\sqrt{(25-9)}=4$ Cm

∴ $AB=8$ cm. ∴ These chords are each 8 cm. long.

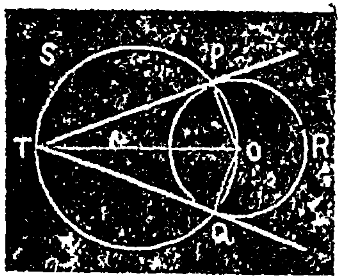
2. See figure Ex. 1.

If AB, CD, EF are each 1.6" long, and $OA=1"$ then OP, OQ, OR are all equal to one another (*Th.* 34.)

Hence they touch a concentric circle whose radius $OP=\sqrt{(OA^2-AP^2)}=\sqrt{(1-.64)}=.6"$

3. See figure Ex 1.

If $OP=2.5$ cm., and $OA=5$ cm., then $AP=\sqrt{(OA^2-OP^2)}=\sqrt{(25-6.25)}=4.33$ cm, nearly. ¶ ∴ $AB=2AP=8.7$ cm, nearly.

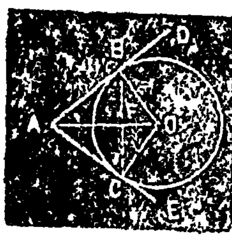


4. Because OP is perpendicular to PT (*Th.* 46), hence if $OP=5"$, $TO=13"$ then $PT=\sqrt{(OT^2-OP^2)}=\sqrt{(169-25)}=12"$.

Draw the figure, measure the tangent and the $\angle POT$, and you will find them equal to 12" and 67° respectively

5. See Figure Ex. 4.

If $OP=.7"$, and $PT=2.4"$, then $OT=\sqrt{(OP^2+PT^2)}=\sqrt{(.49+5.76)}=2.5"$.



6. Let AB and AC be any two intersecting st. lines, and let O be the centre of the circle touching them at the points B and C. *Then the point O must lie on the bisector of the $\angle BAC$.*

Join OB, OC, and OA; then the $\angle$s ABO and ACO are rt. angles (*Th.* 46).

Then in the Δs AOB and AOC, because $OB=OC$, AO is common to both, and the rt. $\angle ABO$=the rt. $\angle ACO$, therefore the triangles are identically equal (*Th* 18). and therefore the $\angle BAO$=the $\angle CAO$ ¶ ∴ The point O lies on the bisector the $\angle BAC$.

7. See figure Ex. 6.

Join *BC* and let it intersect *OA* at the point *F*. *Then shall OA bisect BC at right angles.*

Since $AB=AC$ (*Th.*47,*Cor.*), therefore *ABC* is an isosceles triangle.

Again since *AO* bisects the vertical angle *BAC*,

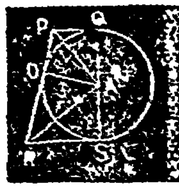
$\therefore$ *AO* bisects the base *BC* perpendicularly.

8. See figure Ex. 4.

Join *PQ* *Then shall the* $\angle PTQ=2$ the $\angle OPQ$.

Because each of the $\angle$s *OPT* and *OQT* is a rt. angle (*Th* 41), therefore the points *O*, *P*, *T* and *Q* are concyclic.

$\therefore$ $\angle OPQ=\angle OTQ$ in the same segment (*Th.* 43.) But $\angle PTQ=2$ $\angle OTQ$ therefore $\angle PTQ=2$ the $\angle OPQ$.



9. Let the two parallel tangents *PQ* and *RS* touching the circle whose centre is *A* at the points *Q* and *S*, be cut by a third tangent *PR* touching the circle at the point *O*. *Then shall PR subtend a right angle at A.*

Join *AP*, *AO* and *AR*. Then because *PQ* and *PO* are two tangents drawn from *P*, therefore the $\angle APO=$the $\angle APQ$ (*Th* 47 *Cor.*)

$\therefore$ The $\angle APO=\frac{1}{2}$ the $\angle QPO$.

Similarly the $\angle ARO=\frac{1}{2}$ the $\angle ORS$.

$\therefore$ The $\angle APO+$the $\angle ARO=\frac{1}{2}$ (the $\angle QPO+$the $\angle ORS$) $=\frac{1}{2}$ of 180° (*Th* 14)$=90^\circ$.

$\therefore$ The third $\angle PAR$ of the $\triangle$ *APR* is also a right angle.

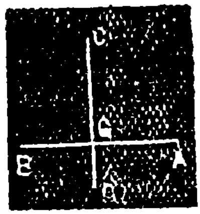
10. Let *QOR* be the diameter of a circle, and let *PX* be the tangent to it at the point *R* *Then shall QOR bisect all chords parallel o PX.*

Draw a chord *AB* parl to *PX* and let it intersect *QR* at the point *C*. Then because

QR is perpendicular to PX, it is also perpendicular to AB, and therefore QR bisects AB at the point C (*Th. 31*).

Similarly it may be proved that QR bisects all chords paralleled to PX.

11. Let AB be a given straight line, and G a given point in it. *It is required to find the locus of the centres of all circles which touch AB at the pt. G.*

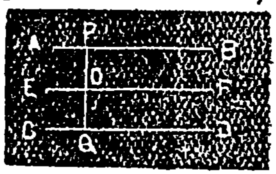


Let C be the centre of any one of such circles. Join CG, then CG is perpendicular to AB at the point G

Similarly if D be the centre of any other of such circles, then DG is perpendicular to AB at the point G.

And since there can be only one perpendicular to AB at the point G, hence the st line CD which is perpendicular to AB at the point G is the required locus.

12 Let AB and CD be any two parallel st. lines, *It is required to find the locus of the centres of all circles touching each of the st lines AB and CD.*



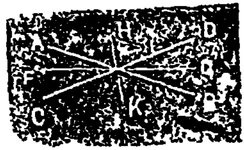
Take any pt. P in AB and draw PQ perpendicular to AB meeting CD at Q.

Then PQ is also perp to CD.

Bisect PQ at O Then a circle described with centre O and radius OP or OQ will touch AB at the point P and CD at Q [*Th. 46*].

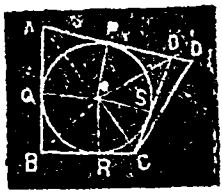
Thus we see that the centre of a circle touching two parallel st. lines is equally distant from them, and therefore the locus of the centres of such circles is a st. line parallel

to the given st lines and drawn midway between them.



13 The centre of any circle which touches two intersecting st lines lies on the bisector of the angle between them (*Ex 6*)

∴ The locus of the centres of all circles which touch two intersecting st lines AB, CD is a pair of st lines FG, HK bisecting the angles between the two given st lines [*Prob 15*]



14 Let $ABCD$ be a quadrilateral circumscribed about the circle $PQRS$ and touching it at the pts P, Q, R and S. *Then shall* $AD+BC$ *be equal to* $AB+CD$

Because from A two tangents AP, AQ are drawn to the circle therefore $AP=AQ$ [*Th. 47. Cor*].

Similarly it may be proved that $DP=DS$, $BR=BQ$ and $CR=CS$ ∴ $AP+DP+BR+CR=AQ+DS+BQ+CS$.

or $AD+BC=AB+CD$

Converse.

If the sum of one pair of opposite sides of a quadrilateral be equal to the sum of the other pair, then a circle can be inscribed in it.

Let $ABCD'$ be such a quadrilateral having $AD'+BC=AB+D'C$.

It is required to prove that a circle can be inscribed in it

Bisect the angles BAD' and ABC by the st. lines AO, BO meeting at the pt O, then O is the centre of the circle touching the sides $D'A$, AB and BC (*Ex 6*)

If this circle does not touch the side $D'C$, then from C

draw CD tangent to the circle, meeting $D'A$ at the pt D. Then $AD+BC=AB+DC$ (*Proved above*)........(i)

Also $AD+BC=AB+D'C$ (*Hyp.*)......(ii)

Subtracting (i) from (ii), we have $AD'-AD=D'C-DC$.

$\therefore DD'=D'C-DC$, which is impossible. [*Ex.* 8, *P.* 34]

$\therefore$ The circle also touches the side $D'C$, and hence a circle can be inscribed in the quadrilateral $ABCD'$.

15. See fig Ex 14.

It is required to prove that AB and CD as well as AD and BC subtend supplementary angle at the centre O.

Join OA, OB, OC, OD, OP, OQ, OR and OS. Then the $\angle AOP=$ the $\angle AOQ$ [*Th.* 47, *Cor.*] and therefore the $\angle AOP$ $=\frac{1}{2}$ the $\angle POQ$.

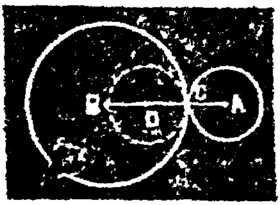
Similarly the $\angle POD=\frac{1}{2}$ the $\angle POS$, the $\angle COR=\frac{1}{2}$ the $\angle SOR$; and the $\angle ROB=\frac{1}{2}$ the $\angle ROQ$.

$\therefore$ The $\angle AOD+$the $\angle COB=\frac{1}{2}$ (the $\angle POQ+$the $\angle POS$ $+$the $\angle SOR+$the $\angle ROQ$) $=2$ rt. angles. (*Th.* 1, *Cor* 2),

Similarly the $\angle AOB+$the $\angle COD=2$ rt. angles.

Exercises on the contact of circle, P. 179

1. Take any two points A and B 2.6" apart. R. F. $\frac{1}{2}$

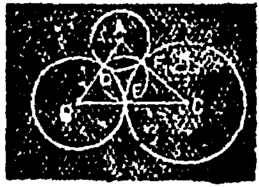


With centres A and B and radii=.9" and 1.7" draw two circles, and notice that they touch externally at a point C in AB such that $AC=.9"$ and $BC=1.7"$. They do so, because the sum of their radii=.9"+1.7"=2.6"=distance between their centres [*Th.* 48. *Cor.* 1].

(ii) From CB cut off $BD=.8"$. With centre D and radius=.9" describe a circle, and notice that this circle touches the circle whose centre is B, internally at the pt C

It does so, because the difference of their radii$=1.7''-.9''$ $=8''=$ the distance between their centres. [*Th. 48, Cor. 2*].

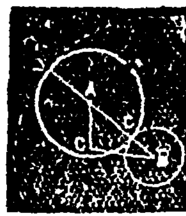
2 Draw a $\triangle\ ABC$ such that BC, CA R. F. $\frac{4}{25}$



AB measure 8 cm, 7 cm, and 6 cm respectively [*Prob. 8*]

With centres A, B and C and radii measuring 2 5 cm, 3 5 cm and 4 5 cm respectively, draw three circles. Then they will touch in pairs at the points D E and F, because $AB=AD+DB=2.5+3\ 5=6$ cm, $BC=BE+EC=3.5+4\ 5=8$ cm, and $CA=CF+FA=4\ 5+2\ 5=7$cm

3 Take any two st. lines CA, CB at rt. R F $\frac{1}{10}$

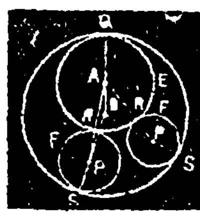


angles to one another, and measuring 6 cm. and 8 cm. respectively. Join AB, *then* ABC *is the required triangle.*

With centre A and radius$=7$ cm. describe a circle cutting AB at D

Because $AB=\sqrt{(AC^2+BC^2)}=\sqrt{(36+64)}$ $=10$ cm. ¶ . If a circle be drawn with centre B to touch the former circle, then the radius of the latter circle$=10-7$ $=3$ cm., or $10+7=17$ cm. as shown in the diagram

4 Take any two points A and B 2 cm R. F. $\frac{1}{5}$



apart. With centres A and B and radii 3 cm. and 5 cm. draw two circles QER and QSS', then they will touch each other internally at the point Q.

Join BA, then BA produced will pass through Q [*Th* 48].

Let P be the centre of the circle touching the circle QSS' internally at S, and the circle QRE externally at R Join AP

and BP, then AP passes through R, and BP produced passes through S [*Th.* 48].

Then because $AP=AR+PR$, and $BP=BS-PS=BS-PR$

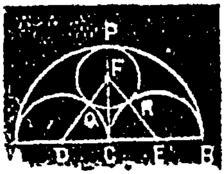
¶ $\therefore AP+BP=AR+PR+BS-PR=AR+BS=3+5=8$ cm.

Similarly if P' be the centre of any other such circle, it can be proved that $AP'+BP'=8$ cm.

$\therefore AP+BP=$ sum of the radii of the given circles, and is therefore constant.

5. Take a st. line $AB=4''$. Divide it R. F. $\frac{1}{3}$

into 4 equal parts at the pts. D, C and E. With centres D, C and E and radii$=1''$, $2''$ and $1''$ respectively, draw semi-circles AQC, APB and CRB.



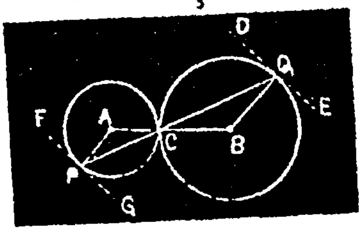
Let F be the centre of the circle touching the semi-circles AQC and CRB externally at the pts. Q and R, and the semi-circle APB internally at the pt. P.

Let x denote the radius of the circle PQR, then $CF=CP-PF=2''-x$ and $DF=DQ+FQ=1''+x$.

Because $DF^2=DC^2+CF^2$, therefore $(1''+x)^2=(1'')^2+(2''-x)^2$

Or $1+2x+x^2=1+4-4x+x^2$, *i. e* $6x=4$ or $x=\frac{2}{3}''$

6. Let two circles whose centres are A and B touch each other at the pt. C. Through C draw any st. line PCQ cutting the circumferences at the pts. P and Q. Join AP and BQ. *Then shall AP be parallel to BQ.*



Join AB, then AB passes through C [*Th.* 48].

Because $AC=AP$, therefore the $\angle ACP=$ the $\angle APC$.

Again because $BC=BQ$, therefore the $\angle BCQ=$ the $\angle BQC$

But the $\angle ACP=$ the $\angle BCQ$, therefore the $\angle APC=$ the $\angle BQC$, and these are alternate angles.

$\therefore AP$ is parallel to BQ [*Th.* 13].

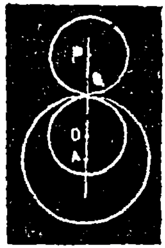
7. See fig. Ex. 6

Draw DQE and FPG tangents to the circles at the pts Q and P respectively. *Then shall FG be parallel to DE.*

Because the $\angle$s DQB and GPA are rt. angles [*Th* 46], and the $\angle APC =$ the $\angle BQC$ [*Ex.* 6]

∴ The rem $\angle GPC =$ the rem. $\angle DQC$, and these are alternate angles.

∴ DE is parallel to FG [*Th* 13].

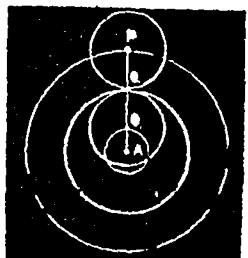


8. Let A be the centre of a given circle of radius a *It is required to find the locus of the centres of all circles (i) which touch it at a given point Q on it, and (ii) which are of a given radius B and touch the given circle*

(i) Let O be the centre of a circle touching the given circle A internally at Q and P that of a circle touching it externally at Q

Join OA and PA, then each of them passes through Q [*Th* 48].

Similarly it can be shown that the st line joining A to the centre of any other such circle passes through Q Hence the st line AQ produced is the required locus



(ii) Let O be the centre of a circle of radius B touching the given circle A internally, and P that of a circle of radius B touching it externally, at any point Q

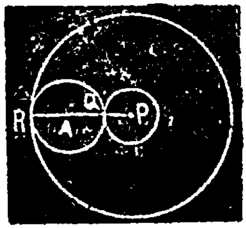
Then $AO = AQ - OQ = a - b$, and

$$AP = AQ + QP = a + b$$

Now because a and b are of constant length, therefore AO and AP which are equal to $a-b$ and $a+b$ respectively are also constant

Hence the required locus is one or other of the

circles whose centre is A, and radius $=(a-b)$ or $(a+b)$ as shown in the diagram.



9. Let A be the centre of a given circle, and P a given point *It is required to draw a circle having P for its centre and touching the given circle.*

Join AP cutting the given circle at the pt Q. With centre P and radius PQ draw a circle, then it will touch the given circle *externally* at Q [*Th.* 48]

Produce PA to cut the given circle at the pt. R.

Then a circle described with centre P and radius PR, will touch the given circle *internally* at R [*Th* 48]

Thus there are *two* solutions of this problem.

10 See Fig (i) Ex. 8

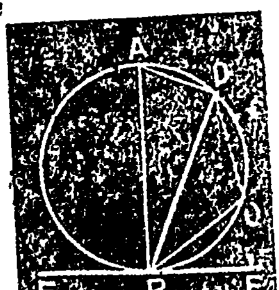
Let A be the centre of the given circle of radius b and let Q be a given point on this circle *It is required to draw a circle of radius a to touch the given circle at the point Q.*

Join AQ and produce it to the pt P such that $PQ=a$. With centre P and radius PQ draw a circle Then this circle will touch the given circle *externally* at the pt Q[*Th.*48].

Again from AQ cut off $OQ=a$ With centre O and radius OQ draw a circle Then this circle will touch the given circle *internally* at the pt Q [*Th* 48]

Thus there are *two* solutions of this problem

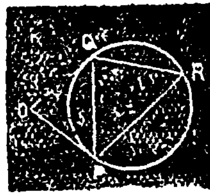
Exercises on Theorem 49, P. 181.



1. If the $\angle FBD = 72°$, then the $\angle BAD$=the $\angle FBD$ [*Th.* 49].=72°.

Since the $\angle$s BAD, BCD together=2 rt. angles [*Th* 40].

∴ The $\angle BCD$=180°-the $\angle BAD$= 180°-72°=108°, and the $\angle$ EBD=the $\angle BCD$ [*Th* 49]=108°



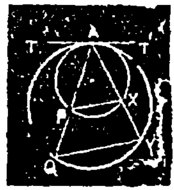
2. Let *PQR* be a given circle, and *OP*, *OQ* two tangents to it from an external pt. *O*. *Then shall OP=OQ*.

Join *PQ*, *PR* and *QR*. Then the $\angle OPQ$ =the $\angle PRQ$ in the alt. segment [*Th*. 49].

Also the $\angle OQP$=the $\angle PRQ$ in the alt. segment [*Th*. 49]

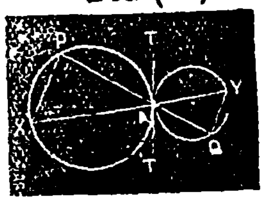
∴ The $\angle OPQ$=the $\angle OQP$, and therefore *OP=OQ*.

Fig (i)



3. Let *APQ*, *AXY* be any two chords drawn from *A* the point of contact of two circles touching each other *internally* as shown in figure (i) and *externally* as shown in fig (ii). Join *PX* and *QY*, *then shall PX be parallel to QY*

Fig (ii)



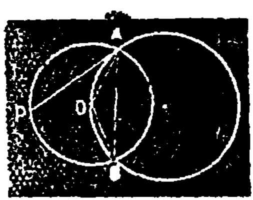
Draw *TAT'* common tangent to the two circles. Then the $\angle TAP$ = the $\angle AXP$ in the alt. segment [*Th*. 49].

and the $\angle TAQ$=the $\angle AYQ$ in the alt. segment fig (i) [*Th*. 49], and the $\angle T'AQ$=the $\angle AYQ$ in the alt. segment fig. (ii) [*Th*. 49]

But the $\angle TAP$=the $\angle TAQ$, in fig (i), and the $\angle TAP$ =the $\angle T'AQ$ in fig (ii)

∴ The $\angle AXP$=the $\angle AYQ$, and therefore *PX* is parallel to *QY* [*Th*. 13]

4 Let *A* and *B* be the points of intersection of two circles one of which passes through *O* the centre of the other circle, and let *AP* be the tangent to the former circle at the pt *A*. Join *OA* and *AB*, *then shall OA bisect the* $\angle BAP$.

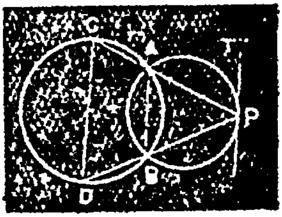


Join OB. Then the $\angle OAP$ = the $\angle OBA$ in the alt. segment [*Th.* 49].

But the $\angle OBA$ = the $\angle OAB$, because radius $OA = OB$.

$\therefore$ The $\angle OAP$ = the $\angle OAB$, and therefore OA bisects the $\angle PAB$.

5. Let two circles APB and $ACDB$ intersect at the pts. A and B, and let there be drawn two straight lines PAC and PBD meeting the circle $ACDB$ at the points C and D. Join CD, and draw PT tangent to the circle APB *Then shall CD be parallel to PT.*



Join AB, then the $\angle TPA$ = the $\angle PBA$ in the alt. segment [*Th.* 49]

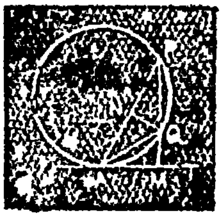
But the $\angle$s PBA, ABD together = 2 rt angles, as also the $\angle$s ACD, ABD together = 2 rt. angles [*Th* 40].

$\therefore$ The $\angle ACD$ = the $\angle PBA$ = the $\angle TPA$.

But the $\angle$s ACD, TPA are alternate angles.

$\therefore PT$ is parallel to CD.

6. Let AT be a tangent to the circle AQP at the pt. A, AP, any chord drawn from A and Q the middle point of the arc AP. From Q draw QM, QN perpendiculars on AT and AP respectively. *Then shall QM be equal to QN.*



Join AQ and PQ. Then because $AQ = PQ$, therefore the $\angle APQ$ = the $\angle QAP$.

Also the $\angle QAT$ = the $\angle APQ$ in the alt. segment [*Th.* 49].

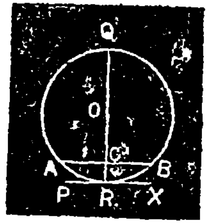
$\therefore$ The $\angle QAT$ = the $\angle QAP$.

Now in the $\triangle$s AQM and AQN, because the $\angle QAM$ = the $\angle QAN$ (*proved*), the $\angle$s at N and M are rt. angles, and the side AQ is common to both, therefore the triangles are identically equal [*Th* 17].

$\therefore QM = QN$.

Exercises on the Method of Limits, P 181

2 Let $AQBR$ be a circle and QR its diameter Draw PRX perpendicular to QR at one of its extremity R. *Then shall PRX be a tangent to the circle at the pt R*

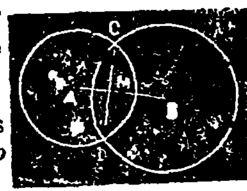


Draw any chord AB parl to PX, then QR is perpendicular to AB ¶ AB is bisected at C [*Th* 31] and this is true *however near C approaches to R*

If C moves up to and coincides with R, then since $AC=CB$, the pts. A and B will also coincide with R, and then the chord will have only one point of contact with the circle at R

∴ Ultimately, the st line drawn perpendicular to the diameter at one of its extremeties is a tangent

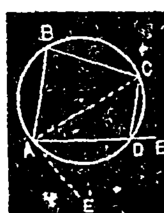
3 Let two circles whose centres are A and B intersect at the pts C and D, Join AB and CD, and let them intersect at the pt M, then AB bisects CD at rt angles at the pt. M. (*Hyp*), and this is true however *C and D approach near to each other*



If C and D come *infinitely* near to each other, then since $CM=DM$, the pts C and D will ultimately coincide with M and the circles will then touch each other at the pt M

∴ Ultimately, the straight line joining the centres of two circles touching each other, passes through the pt of contact.

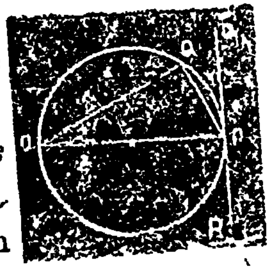
4 Let $ABCD$ be a cyclic quadrilateral and let the side AD be produced to E Then the exterior $\angle CDE$ = interior and opposite $\angle ABC$ [*Ex* 5, *Page* 163] And this is true *however near D approaches to A*



If D moves up to and coincides with A the chord CD will ultimately become the

chord CA and the st. line DE will become the tangent AE'. Hence the $\angle CDE$ will ultimately coincide with the $\angle CAE'$.

$\therefore$ Ultimately the $\angle CAE'$ = the $\angle ABC$ in the alternate segment.



5. Let OQP be a circle and OP its diameter. Join OQ, PQ. Then the $\angle OQP$ is a right angle [*Th* 41], and this is true *however near Q approaches to P.*

If Q moves up to and coincides with P, then the chord OQ will become the diameter OP, and the chord PQ will become the tangent PR, also the $\angle OQP$ will ultimately coincide with the $\angle OPR$.

$\therefore$ *The tangent at any point of a circle is perpendicular to the diameter drawn to the point of contact.*

Exercises on Common Tangents, P. 187.

1. With the pts. A and B as centres and radii $=1.4''$ and $1''$ respectively draw two circles, making AB successively $=1''$, $2.4''$, $.4''$ and $3''$, and notice that the circles *intersect* in the first case, *touch externally* in the second case, *touch internally* in the third case and *lie one beyond the other* in the fourth case as shown in the accompanying diagrams.

As is clear from the diagrams; we can draw *two direct* tangents in the first case, *three* tangents in the second case, only *one* tangent in the third case, and *four* tangents two *direct* and two *transverse* in the fourth case.

Fig. (i).

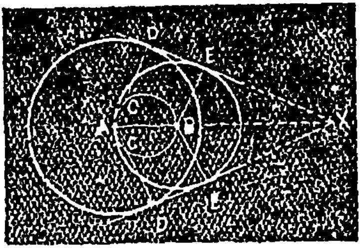


Fig (ii)

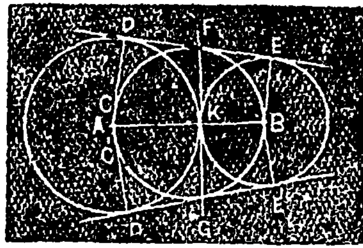


Fig (iii)

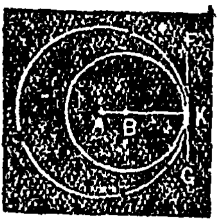
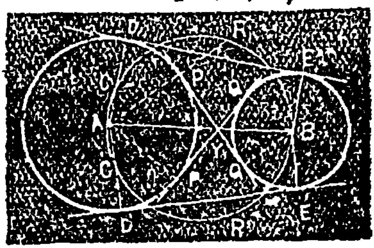


Fig (iv)



(i) Upon the diameter AB draw a circle, and with centre A and radius = the difference of the two given radii $(1.4''-1'') = .4''$ cut this circle at the pts C and C'. Join AC and AC', and produce them to cut the larger circle at the pts D and D'. From B draw BE, BE' radii of the smaller circle *parl* to AD and AD' respectively. Join DE, $D'E'$, then these are the *direct common* tangents. *See figures* (*i*), (*ii*) *and* (*iv*).

(ii) Let the two circles touch one another at the pt K Draw FKG perp to AB, then FG is a common tangent to the given circles at their point of contact *See figures* (*ii*) *and* (*iii*).

(iii) Upon the diameter AB describe a circle, and with centre A and radius = the sum of the two give

radii $(1.4''+1'')=2.4''$ cut this circle at the pts. R and R'. Join AR, AR', and let them cut the larger circle at the pts. P and P' From B draw BQ, BQ', radii of the smaller circle parallel *but in opposite sense* to AP and AP' respectively Join PQ, PQ' then these are *transverse* common tangents. *See figure* (iv)

2 Take any two points A and B $2''$ apart, and with centres A and B and radii $=2''$ and $8'$ respectively draw two circles, *and notice that they intersect one another*

Then proceed as in Ex. 1, figure (i)

Measure DE and $D'E'$ and you will find them each$=1\ 6''$.

By *calculation* $DE=BC=\sqrt{(AB^2-AC^2)}=\sqrt{(4-1.44)}$ $=1.6''$.

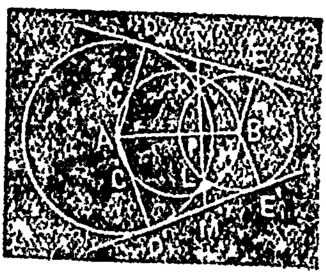
3 Take any two points A and B $1\ 8''$ apart, and with centres A and B and radii$=6''$ and $1\ 2''$ respectively draw two circles *and notice that they touch each other externally*.

Then proceed as in Ex. 1. figure (ii).

Measure DE and $D\ E'$, and you will find them each$=1\ 7''$.

By *calculation* $DE=BC=\sqrt{(AB^2-AC^2)}=\sqrt{(3.24-36)}=$ $1\ 7''$ *nearly*.

4. Take any two points A and B $2\ 1''$ apart With centres A and B and radii$=1\ 7''$ and $1''$ respectively draw two circles *and notice that they intersect one another*.



Then proceed as in Ex 1, fig.(i).

Measure DE and $D'E'$ and you will find them each$=1\ 98''$

By *calculation* $DE=BC=\sqrt{(AB^2-AC^2)}=\sqrt{(4\ 41-49)}=$ $1\ 98''$.

(ii) Let the two circles intersect at the pts. K, and L Join KL, and let it intersect AB at the pt. P Measure KL and you will find it$=1.6''$.

By *calculation*, if $KP=x$, then $AP=\sqrt{AK^2-KP^2}=\sqrt{2.89-x^2}$, and $BP=\sqrt{BK^2-KP^2}=\sqrt{1-x^2}$,

$\therefore AB=AP+BP=\sqrt{2.89-x^2}+\sqrt{1-x^2}$

But $AB=2.1$, therefore $\sqrt{2.89-x^2}+\sqrt{1-x^2}=2.1$, whence $x=.8$

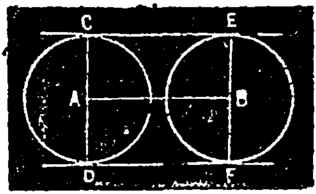
$\therefore KL=2x=1.6''$.

(iii) Produce KL both ways to meet DE, $D'E'$ at the points M and M' respectively Measure and notice that $DM=ME$, and $D'M'=E'M'$

5 Take any two points A and B 3" apart. With centres A and B, and radii $=1.6''$ and $.8''$ respectively draw two circles, and notice that one circle lies wholly outside the other

Then Proceed as in Ex 1, fig. (iv).

6. With any two pts A and B as centres and any convenient radius draw two equal circles. Join AB, and draw CAD, EBF diameters of these circles each perp to AB.



Join CE and DF, then these are the direct common tangents

7. (i) See Ex. 1, Fig (i)

$DE=BC=\sqrt{(AB^2-AC^2)}$, and $D'E'=BC'=\sqrt{(AB^2-AC'^2)}$

But $AC=AC'$, therefore $DE=D'E'$.

(ii) See Ex 1 fig (iv)

Join BR and BR', then because the $\angle s$ at P and R are right angles [*Ths* 46 & 41] $\therefore BR$ is parallel to PQ

And since BQ is parl to PR, therefore $PQBR$ is a parallelogram.

$\therefore PQ=BR=\sqrt{(AB^2-AR^2)}$

Similarly $P'Q'=BR'=\sqrt{(AB^2-AR'^2)}$

But $AR=AR'$, therefore $PQ=P'Q'$.

8. (i) See Ex 1, figure (i).

Let DE, $D'E'$ meet at the point X. Join AX and BX, *then they shall be in one st. line.*

Because the rt angled $\triangle$s ADX and $AD'X$ are identically equal [*Th.* 47, *Cor.*], therefore the $\angle AXD$ = the $\angle AXD'$, and therefore AX bisects the $\angle DXD'$.

Similarly BX bisects the $\angle EXE'$.

But the $\angle DXD'$ is the same as the $\angle EXE'$.

$\therefore$ AX and BX are in the same st. line.

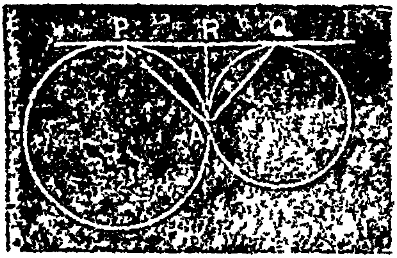
(ii) See Ex. 1, figure (iv)

Let PQ, $P'Q'$ meet at the pt Y. Then as shown above AY bisects the $\angle PYP'$, and BY bisects the $\angle QYQ'$.

But the $\angle PYP'$ = the $\angle QYQ'$, therefore AY and BY are in the same st line.

Hence the *direct* as well as the *transverse* common tangents intersect on the line of centres.

9. Let PQ be a common tangent to two circles whose centres are B and C, and which touch each other externally at A, *then shall PQ subtend a rt. angle at A*



Let the common tangent to the two circles at A meet PQ at R.

Then since RP, RA are two tangents from R to the circle APD

$\therefore$ $RP = RA$ [*Th.* 47, *Cor.*], and therefore $\angle RPA = \angle RAP$.

Similarly $RQ = RA$, and therefore $\angle RQA = \angle RAQ$

$\therefore$ $\angle PAQ = \angle APQ + AQP$.

$\therefore PAQ$ is a right angle [*Th.* 16. *Inf* 4].

Notes on loci at the foot of p. 188

(i) The locus of the centres of circles which passes through two given points is *a straight line bisecting the st line joining the two given points perpendicularly.* (*Ex.* 4, *P.* 147).

(ii) The locus of the centres of circles which touch a given straight line at a given point is *a straight line perpendicular to the given st line at the given point.*(*Ex. 11,P* 177)

(iii) The locus of the centres of circles which touch a given circle at a given point is *the straight line joining the centre of the given circle with the given point* [*Ex* 8 (i) *P* 179]

(iv) The locus of the centres of circles which touch a given straight line, and have a given radius is *one or other of the two straight lines parallel to the given straight line on either side of it and at a distance of the given radius from it.*

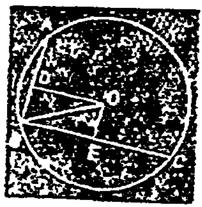
(v) The locus of the centres of circles which touch a given circle and have a given radius is *a concentric circle whose radius=the sum or difference of the two radii* [*Ex.* 8 (ii). *P* 179]

(vi) The locus of the centres of circles which touch two given lines is *a pair of straight lines bisecting the angle between the two straight lines.* [*Ex* 13, *P* 177].

If the given st. lines are parallel, then the locus is the *straight line parallel to the given straight lines and midway between them* [*Ex* 12 *P.* 177]

Exercises on the construction of circle. Page 189.

1 Let *A* *B* and *C* be any three given points. *It is required to draw a circle passing through them*



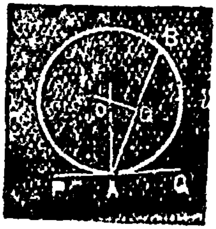
(i) Centre of a circle passing through the pts. *A* and *B* lies on the st line *DO* bisecting *AB* perpendicularly, [*Note* (i) *P* 188]

(ii) Centre of a circle passing through the pts *B* and *C* lies on the st line *EO* bisecting *BC* perpendicularly (*Note* ii), *P* 188]

∴ The point where the st. lines DO and EO intersect. satisfies both the conditions and is, therefore, the centre of the circle passing through A, B and C.

With centre O and radius OB describe the circle, then it will pass through A and C.

2. (i) If a circle touches a st. line PQ at a point A, its centre lies on a straight line AD perpendicular to PQ at A.[*Note* (*II*) *P*. 188]

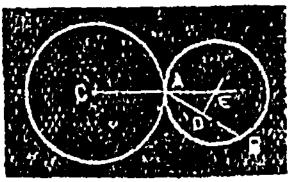


(ii) If a circle passes through two given points A and B, its centre lies on the straight line CD bisecting AB perpendicularly. [*Note* (i) *P*. 188].

∴ The pt. D where these two straight lines intersect, satisfies both the conditions, and is, therefore the required centre.

With centre D and radius DA describe the circle, then it will pass through the point B, and touch the straight line PQ at A.

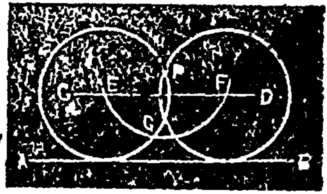
3. (i) If a circle touches a given circle whose centre is C at the point A, then its centre lies on the straight line CA.[*Note* (iii) *P* 188].



(ii) A circle passing through the points A and B has its centre on the st. line DE bisecting AB perpendicularly. [*Note* (i) *P*. 188].

∴ The centre of the circle which touches the given circle at the point A and passes through the point B is the point E where the st lines DE and CA intersect. With centre E and radius EA describe the required circle.

R. F $\frac{1}{4}$



4 Let P be any point 4·5 cm distant from a given st. line AB. *It is required to draw two circles of radius 3·2 cm to pass through P and to touch the st line AB.*

(i) Locus of the centres of circles of radius 3·2 cm. which touch the given st line AB is the st line CD parallel to AB and at a distance of 3·2 cm from it [*Note* (iv), *P* 188]

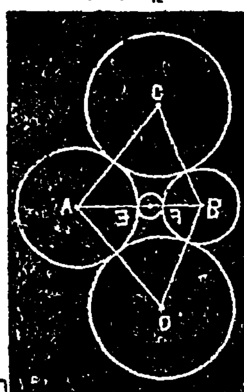
(ii) Locus of the centres of circles of radius 3·2 cm which passes through the pt P, is a circle EGF whose centre is P and radius=3·2 cm

Let the circle EGF cut the st line CD at the pts E and F. Then these points satisfy both the conditions and hence are the centres of the required circles

With centres E and F and radius 3·2 cm describe the circles and notice that they satisfy the given conditions.

5 Draw two circles of radius 3 cm and 2 cm respectively, and having their centres A and B 6 cm apart. *It is required to draw a circle of radius 3·5 cm to touch each of the given circles externally.*

R F $\frac{1}{4}$



(i) Locus of the centre of such a circle is a circle whose centre is A and radius=(3+3·5)=6·5 cm [*Note* (v), *P*, 188]

(ii) Locus of the centre of such a circle is also a circle whose centre is B and radius=(2+3.5)=5.5 cm [*Note* (v), *P*.188]

Let these two circles intersect at the pts. C and D, then these are the centres of the required circles.

With centres C and D and radius 3 5 cm, draw two circles and notice that they satisfy the given conditions. Thus there are *two* solutions

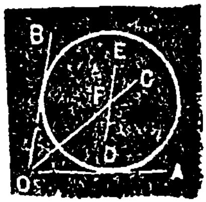
The smallest circle which touches the circles A and B *externally* is the circle whose centre lies on AB midway between the pts. E and F where the given circles cut AB.

$\therefore$ Its radius$=\frac{1}{2}EF=\frac{1}{2}(AB-(AE+BF)=\frac{1}{2}[6-(3+2)]$.

$=.5$ cm

6 (i) If a circle touches two st. lines OA and OB making an angle of 76° between them, then its centre lies on the st. line OC bisecting the angle AOB (*Note* (vi), *P.* 188).

R. F. $\frac{1}{4}$



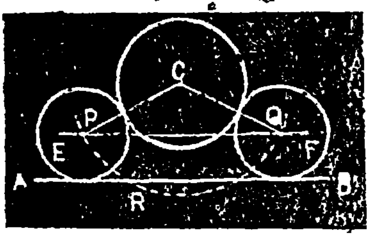
(ii) Locus of the centres of circles of 1 2" radius and touching the st line OB, is a st line DE parallel to OB and at a distance of 1 2 from it. (*Note* (iv), *P.* 188)

Hence the pt. F, where the st line DE and OC cut each other, is the centre of the circle whose radius=1.2" and which touches the st lines OA and OB

With the pt. F and radius=1 2" draw the required circle.

R F. $\frac{1}{5}$

7 Let AB be a given st. line, and let a pt. C. 5 cm. distant from the st line AB, be the centre of a given circle of radius 3 5 cm *It is required to draw two circles of radius 2 5 cm to touch the given circle and the given st line AB.*



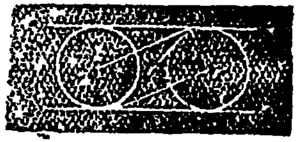
(i) Locus of the centre of a circle of radius 2 5 cm. touching the st. line AB, is a st line EF parallel to AB and at a distance of 2.5 cm. from it (*Note* (iv), *P.* 188).

(ii) Locus of the centre of a circle of radius 2 5 cm. touching the given circle, is a circle PRQ whose centre is C. and whose radius $=(3\ 5+2\ 5)=6$ cm (*Note* (v), *P.* 188)

The pts P and Q where the circle PRQ cuts the st line EF, are the centres of the required circles

With centres P and Q and radius 2 5 cm draw the circles

8. Let AB and CD be any two parallel st lines, and EF any other transversal cutting AB and CD at the pts. E and F respectively.



It is required to draw a circle to touch these three st lines

(i) Locus of the centres of circle touching the st lines AB and EF is one or other of the st lines EG, EH bisecting the angles AEF and BEF respectively, (*Note* (iv) *P* 188)

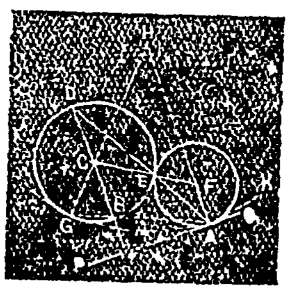
(ii) Locus of the centres of circles touching the st lines CD and EF is one or other of the st lines FG, FH bisecting the angles CFF and DFE respectively (*Note* (vi), *P.* 188).

Hence the points G and H, where these four st lines intersect, are the centres of the required circles.

Draw the required circles, then the radius of each of them $=\frac{1}{2}$ the perpendicular distance between AB and CD (*Ex* 9, *P* 177).

$\therefore$ The two circles are equal.

9. Let C be the centre of a given circle, and A a given point in a given st line PQ. *It is required to draw a circle to touch the given circle and the given st line PQ at the pt A*



At A draw AF perpendicular to PQ then the centre of the required circle lies on AF.

Draw BD the diameter of the given circle perp. to PQ. Join AD cutting the circumference of the given circle at E. Join CE, and produce it to cut AF at the pt F. *Then F is the centre of the required circle.*

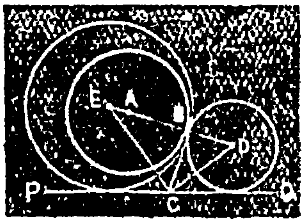
Proof—Because BD and AF are both perp to PQ, therefore BD is parallel to AF.

$\therefore$ The $\angle EAF$ = the alt $\angle CDE$
= the $\angle CED$ [for $CD = CE$]
= the $\angle AEF$

$\therefore$ $AF = EF$.

$\therefore$ If a circle be described with centre F and radius FA, it will touch the st. line PQ at A and the given circle at E.

Note—Join AB, and produce it to cut the circumference of the given circle again at the point G. Join GC, and produce it to meet the st. line AF at H *Then H is the centre of another such circle.* Complete the proof as given above.



10. Let B be a given point on the circumference of the given circle whose centre is A, and let PQ be a given st. line. *It is required to draw a circle to touch the st. line PQ and the given circle at B.*

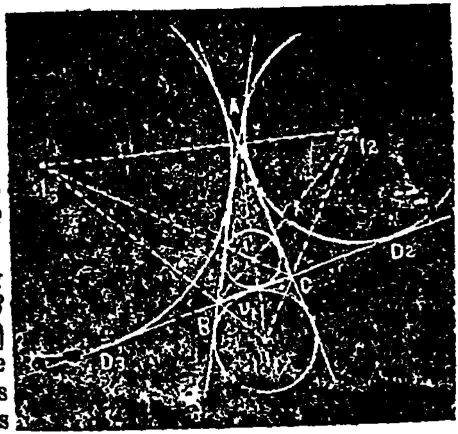
Join AB, and draw BC perpendicular to AB meeting PQ at C Then BC shall be the common tangent to the two circles.

(i) Centre of the required circle lies on the st. line AB produced [*Note* (III), *P*. 188]

(ii) Also the centre of the required circle which touches the st. lines BC and PQ lies on one or other of the st. lines CD, CE which bisect the $\angle$s QCB, PCB respectively. [*Note* (VI), *P*. 188].

∴ The points D and E, where AB produced cuts CD and CE, are the centres of the required circles. Draw the circles as shown in the diagram

11. Let AB, BC and CA be three given straight lines. *It is required to draw circles touching each of these st. lines*



(i) Locus of the centres of circles touching the st. lines BA and BC is one or other of the st. lines BI_2, BI_3 bisecting the angles between AB and BC [*Notes* (*VI*) *P.* 188]

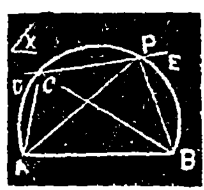
(ii) Locus of the centres of the circles touching the st. lines CA and CB is one or other of the st. lines CI_1, CI_3 bisecting the angles between CA and CB [*Note* (VI), *P.* 188]

Hence the pt. I, I_1, I_2, and I_3 where the st. lines BI_1, BI_2, CI_1 and CI_3 intersect, are the centres of the required circles. Draw the circles as shown in the diagram above.

Thus we see that there can be drawn *four* circles to touch each of the three given st. lines.

Exercises on Prblem 24 P. 191

1. Let X be a given angle, AB the given base, and DE a given st. line. *It is required to describe a triangle on the base AB having its vertical angle = X, and vertex on the st. line DE*



(i) On AB describe a segment ACB which shall contain an angle = the $\angle X$ [*Prob.* 24] Then the vertex of the required triangle lies on the arc ACB

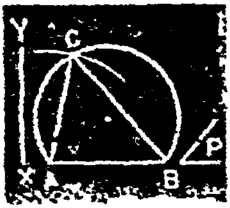
(ii) Also the vertex lies on the st. line DE.

∴ The points C, P where the st. line DE cuts the arc ACB, denote the vertex of the required triangle. Join AC, BC and AP, BP. Then ABC and ABP are two such triangles.

2. Let AB be the given base, and P the given vertical angle.

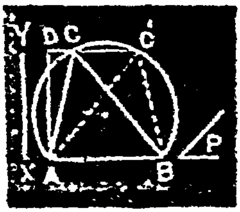
On AB describe a segment ACB which shall contain an angle = the $\angle P$ [Prob 24]. Then the vertex of the triangle whose base is AB and vertical angle = the $\angle P$, lies on the arc ACB.

(i) Let XY be the length of one of the sides of the triangle. With centre A and radius $= XY$ draw an arc, then the vertex of the required triangle also lies on this arc.



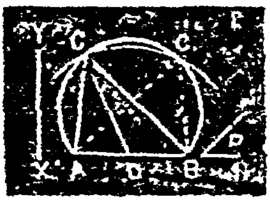
∴ The pt C where this arc cuts the given arc ACB, denotes the required vertex. Join AC, BC. Then ABC is the required triangle.

(ii) Let XY be the length of the given altitude. From A draw AD perpendicular to the st. line AB and make $AD = XY$. From D draw DC parallel to AB. *Then the vertex of the required triangle lies on the st. line DC.*



∴ The points C and C' where the st. line DC cuts the arc ACB, denote the vertex of the required triangle. Join AC, BC and AC', BC'. Then ABC and ABC' are the required triangles.

(iii) Let XY be the length of the median which bisects the base AB. Bisect AB at D. With centre D and radius $= XY$ draw an arc. *Then the vertex of the required triangle lies on this arc.*



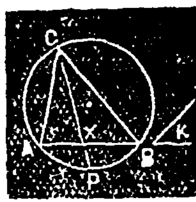
. The points C and C' where this arc cuts the given arc ABC, denote the vertex of the required triangle Join AC, BC and AC', BC'. Then ABC and ABC' are two such triangles constructed

(iv) Let the pt D denote the foot of the perpendicular from the vertex on the base AB *Then the vertex of the required triangle lies on the st line DC which is perpendicular to the st line AB at the pt D.*

Hence pt C where DC cuts the arc ACB is the vertex of the required triangle Join AC, BC Then ABC is the required triangle

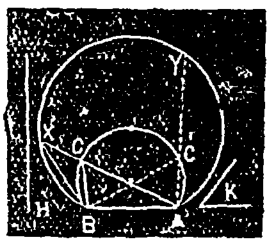
Y **N. B** —For two solutions, see note given at the top of page 191 of your text book

3. *Complete the construction as given in your text book*



b Since the arc AP=the arc PB
∴ The $\angle ACP$ =the $\angle BCP$ [*Th* 43]¶
∴ The st line CP is the bisector of the vertical $\angle ACB$ which is equal to the given $\angle K$
ABC is the required triangle

4. *Complete the construction as given in your text book.*



Because the $\angle ACB = \angle K$, and the $\angle AXB = \frac{1}{2} \angle K$

∴ the $\angle CBX = \angle ACB - \angle AXB$ [*Th* 16]$= \frac{1}{2} \angle K$

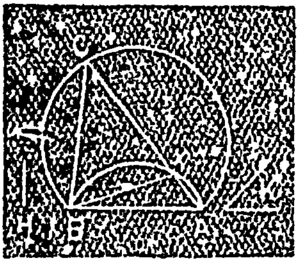
The $\angle CBX$=the $\angle CXB$, and therefore $CB = CX$

The sum of the other two sides AC, CB of $\triangle ABC = AC + CB = AC + CX = AX = H$ Also the vertical $\angle ACB$ = the given $\angle K$ ¶ ABC is the required triangle

N. B.—Let Y be the other point where the circle drawn with centre A and radius H cuts the greater segment

Join AY cutting the smaller segment at the pt C'. Join BC'. Then ABC' is another such triangle.

5. Let AB be the given base, K the given vertical angle, and H a st line equal to the difference of the other two sides. *It is required to construct the triangle*



On the st. line AB describe a segment ACB containing an angle=the $\angle K$, also describe another segment AXB containing an angle$=90°+\frac{k}{2}$ With centre A and radius H, describe an arc cutting the latter segment at the pt. X. Produce AX to meet the former segment at the pt. C, and join BC Then ABC is the required triangle.

Because the $\angle CXB=180°$-the $\angle AXB=180°-(90°+\frac{k}{2})=90°-\frac{k}{2}=90°-\frac{c}{2}$.

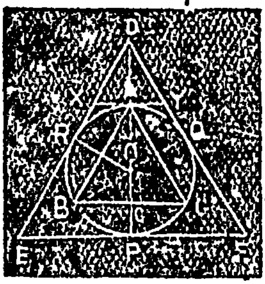
Also the $\angle CBX$ = the $\angle AXB$ - the $\angle C$ [*Th.* 16]$=(90+\frac{c}{2})-C=90°-\frac{c}{2}$.

∴ The $\angle CXB$=the $\angle CBX$, and therefore $CB=CX$

∴ The difference of the sides AC, CB of the $\triangle ABC=AC-CB=AC-CX=AX=H$, also the vertical $\angle ACB$=the $\angle K$, ¶ ∴ ABC is the required triangle.

Exercises on Circles and Triangles, Page 198.

R. F. $\frac{1}{7}$



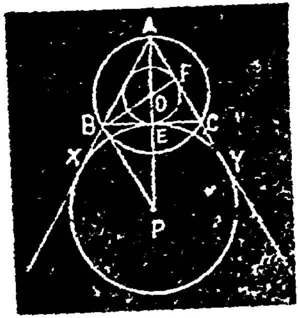
1. With any point O as centre and radius=5 cm. describe a circle. *It is required to inscribe and circumscribe an equilateral triangle in and about it*

(1) At any point A on the circumference draw a tangent XAY to the circle [*Prob* 22], and make the $\angle s\ XAB$, YAC each$=60°$. Join BC. Then ABC is the

required inscribed equilateral triangle. [*Prob.* 28]

(ii) Draw a radius OP perp to BC and make the $\angle POQ$, POR each$=120°$. Draw EF, FD and DE tangents to the circle at the pts P, Q and R. Then DEF is the required *circumscribed* equilateral triangle. [*Prob.* 29].

R P $\frac{3}{10}$



2. Take a st line $BC=8$cm with centres B and C, and radius$=8$cm draw two arcs cutting at the pt. A Join AB, AC. Then ABC is the required equilateral triangle.

Bisect the $\angle s$ A and B by st. lines AE, BF intersecting at O, then O is the centre of the inscribed circle [*Prob.* 26]

Because AE, BF bisect the sides BC, AC perpendicularly, therefore their point of intersection O is the centre of the circumscribed circle [*Prob* 27].

Produce AB to X, and bisect the $\angle CBX$ by a straight line meeting AE produced at P, then P is the centre of the e-inscribed circle touching the side BC [*Prob* 27].

Proof—Because the $\angle EBP=\frac{1}{2}$ the $\angle EBX=60°$

$\therefore$ In the $\triangle s$ ABE, and PBE, the rt $\angle AEB=$the rt. $\angle PEB$, the $\angle ABE=$the $\angle PBE=60°$, and the side BE is common to both, therefore the triangles are identically equal [*Th.* 17] and therefore $PE=AE$.

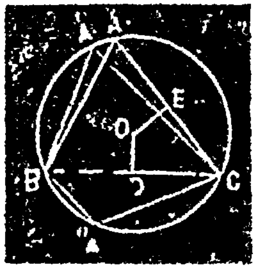
Again because $OE=\frac{1}{3}$ AE, and $OA=\frac{2}{3}AE$ [*Prob. III*, *Cor* *P.* 97] therefore $OA=2OE$, and $PE=AE=3$ OE.

$\therefore$ The circum-radius and ex-radius are respectively double and treble of the in-radius.

$AE=\surd(AB^2-BE^2)=\surd(64-16)=6.9$ cm. $\therefore$ $OE=2.3$ cm. $OA=4.6$ cm, and $PE=6.9$ cm

Measure them, and notice that they correspond with the above lengths.

3 Take a st. line $BC=2.5''$, and make the $\angle s\ CBA,\ BCA=66^\circ$ and 50° respectively, and let their arms meet at A.



Bisect BC, AC at the points D and E, and draw DO, EO perps. to BC and AC respectively, meeting at the point O With centre O and radius OC draw a circle, then it will pass through the pts A and B (*Prob.* 25).

Measure OC, and notice that it$=1.39''$.

In case (ii) and (iii) the same construction is to be made as in case (i), only the $\angle s\ B$ and C are to be made$=72^\circ$ and 44° in case (ii), and $=41^\circ$ and 23° in case (iii).

Value of the vertical $\angle A$.

$$=180^\circ-(B+C)\ (Th.\ 16)$$
$$=180^\circ-(66^\circ+50^\circ)=64^\circ \text{ in case (i).}$$
$$=180^\circ-(72^\circ+44^\circ)=64^\circ \text{ ,, . (ii)}$$
$$=180^\circ-(41^\circ+23^\circ)=116^\circ \text{ ,, ,, (iii).}$$

Now because the base BC is of the same length in all the three cases, and the vert $\angle A$ in case (i) = the vert. $\angle A$ in case (ii), and supplementary of the vert $\angle A$ in case (iii) therefore the circum-circles of the $\triangle ABC$ are equal in the three cases ¶ $\therefore$ Their circum-radii are also equal.

4 See figure Ex. 1

Let ABC, DEF be the inscribed and circumscribed equilateral triangles in and about a circle of 4 cm. radius Draw AG perp. to BC Then AG bisects BC, and therefore passes through the centre O

(i) Because $AG=\frac{3}{2}\ AO$ [*Prob* III *Cor P.* 97] therefore $AG^2=\frac{9}{4}\ AO^2$. Also $AG^2=AB^2-BG^2=AB^2-\frac{1}{4}AB^2=\frac{3}{4}AB^2$

$\therefore \frac{3}{4}\ AB=\frac{9}{4}\ AO^2$, and therefore $AB^2=3\ AO^2=3\times 16=48$.

$\therefore AB=4\sqrt{3}=6.9$ cm.

Measure AB, and notice that it$=6.9$ cm.

(ii) Because $BC=AB=6\ 9$ cm, and $AG=\frac{3}{2}\ AO=6$ cm

$\therefore$ The area of the $\triangle ABC=\frac{1}{2}\ AG\ \ BC=\frac{1}{2}\times 6\times 4\sqrt{3}$ cm $=12\sqrt{3}$, or 20 78 sq cm

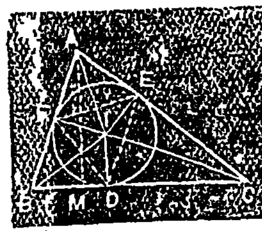
(iii) Because O is the centroid of the $\triangle\ DEF$, therefore $OD=2\ OP=8$ cm, and $DP=3\ OP=12$ cm [*Prob III, Cor. P. 97*]

Again because $EP=\sqrt{OE^2-OP^2}=\sqrt{64-16}=4\sqrt{3}$ cm, therefore $EF=8\sqrt{3}$ sq cm.

The area of the $\triangle DEF=\frac{1}{2}\ EF.\ DP=\frac{1}{2}\times 12\times 8\sqrt{3}=48\sqrt{3}$ sq cm

$\therefore$ The $\triangle ABC=\frac{1}{4}$ of the $\triangle DEF$

5. Let ABC be a triangle, and let the bisectors of the $\angle s\ A$ and B meet at the pt I, then I is the centre of the inscribed circle Draw ID, IE and IF perps to BC, CA and AB respectively Then each of them is the radius of the inscribed circle and therefore each$=r$



(i) $\triangle\ IBC=\frac{1}{2}ID\ \ BC=\frac{1}{2}\ ar$, $\triangle ICA=\frac{1}{2}IE.\ AC=\frac{1}{2}\ br$, and $\triangle\ IAB=\frac{1}{2}\ IF\ \ AB=\frac{1}{2}\ cr$

$\therefore\quad \triangle\ ABC=\triangle IBC+\triangle ICA+\triangle IAB$

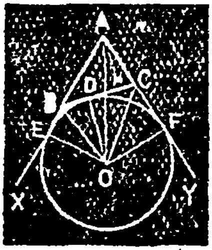
$$=\tfrac{1}{2}ar+\tfrac{1}{2}br+\tfrac{1}{2}cr=\tfrac{1}{2}(a+b+c)\ r.$$

(ii) If $AB=9$ cm, $BC=8$cm, and $AC=7$cm, then ID will be found to be 2 24 cm.

$\therefore\ \triangle\ ABC=\frac{1}{2}\ (a+b+c)\ r=\frac{1}{2}\ (9+8+7)\times 2\ 24=26\ 8$, sq cm.

Now draw AM perp. to BC, then it will be found to be$=6\ 7$ cm, ¶ $\therefore\ \triangle\ ABC=\frac{1}{2}AM.\ BC=\frac{1}{2}\times 6\ 7\times 8=26.8$ sq cm

Thus we see that the formula given in case (i) is true.



6. Let ABC be a triangle. Produce the sides AB, AC to any pts X and Y. Bisect the $\angle s$ CBA and BCY by the st lines meeting at D Then O is the excentre of the triangle ABC opposite to A [*Prob.* 27]. Draw OD, OE and OF perps. to BC, AX and AY respectively Then each of them is the radius of the escribed circle, and therefore each$=r$.

(i) $\therefore$ $\triangle$ $ABO=\frac{1}{2}$ AB. $OE=\frac{1}{2}\,cr^1$, $\triangle$ $ACO=\frac{1}{2}$ AC. $OF=\frac{1}{2}$ br^1, and $\triangle$ $BCO=\frac{1}{2}$ BC $OD=\frac{1}{2}\,ar^1$.

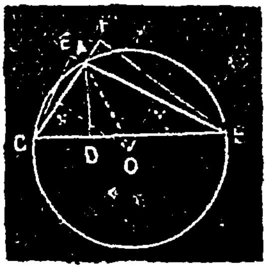
$\therefore$ $\triangle$ $ABC=\triangle$ $ABO+\triangle$ $ACO-\triangle$ $BCO=\frac{1}{2}\,cr^1,+br^1-\frac{1}{2}$ $ar^1=\frac{1}{2}\,(c+b-a)r^1$.

(ii) If $BC=5$ cm, $AC=4$ cm, and $AB=3$ cm, then OD will be found to be$=6$ cm.

$\therefore$ $\triangle$ $ABC=\frac{1}{2}\,(c+b-a)\,r^1=\frac{1}{2}\,(4+3-5)\times 6=6$ sq cm.

Now draw AM perp to BC, then it will be found to be $=2\,4$ cm Therefore area of the $\triangle$ $ABC=\frac{1}{2}$ AM. $BC=\frac{1}{2}\times 2.4$ $\times 5=6$ sq cm.

Thus we see that the formula given in case (I) is true.



7. (i) Draw a triangle ABC such that a $=6\,3$ cm, $b=3$ cm, and $c=5\,1$ cm [*Prob.* 8] Bisect the sides AC, AB at the points X, Y and draw XO, YO perps. to AC, AB meeting at the point O. *Then O is the circum-centre of the* $\triangle ABC$. [*Prob.*25]

Measure OA and you will find it$=3\,2$ cm. nearly

(ii) From A, B, C draw AO, BF, and CE perps. to BC. CA. and AB respectively.

Measure them. and you will find $AD=2.4$ cm., $BF=5\,04$ cm. and $CE=2.96$ cm.

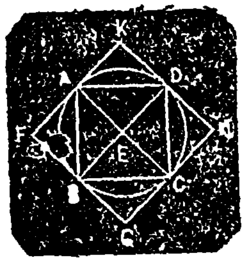
If AD, BF and CE be represented by p^1 p^2 and p^3 then $\frac{bc}{2p^1}=\frac{3\times 5.1}{2\times 2.4}=3.2$ cm. nearly, $\frac{ca}{2p^3}=\frac{5\,1\times 6\,3}{2\times 5.04}=3.2$ cm. nearly

and $\frac{ab}{2p^3} = \frac{6\ 3 \times 8}{2 \times 2\ 96} = 3\ 2$ cm nearly.

$\therefore$ Circum-radius $= 3.2$ cm. $= \frac{bc}{2p^1} = \frac{ca}{2p^2} = \frac{ab}{2p^3}$

Exercises on circles and squares, P. 199.

R F $\frac{1}{?}$



1. Draw a circle of 1.5" radius, and draw in it any two diameters AC, BD intersecting at rt angles to one another at the point E. Join AB, BC, CD and DA. *Then $ABCD$ is the required inscribed square*

By *calculation* $AB = \sqrt{AE^2 + BE^2} = \sqrt{2AE^2} = AE\sqrt{2} = 1\ 5\sqrt{2} = 2\ 12''$.

Measure AB, and you will find it$=2.12''$

Area of the square $ABCD = \frac{1}{2}\ AC.\ BD = \frac{1}{2} \times 3 \times 3 = 4.5$ sq in,

2 See fig Ex 1.

At the points A, B, C and D draw tangents meeting one another at the points F, G, H, and K. *Then $FGHK$ is the required circumscribed square*

Because the squares $AEBF$, $BECG$, $CEDH$ and $DEAK$ are respectively double of the triangles ABE, BEC, CED and DEA taken in order [*Th.* 21].

$\therefore$ The whole square $FGHK$ is double of the whole square $ABCD$.

3 See fig Ex 1.

(i) Describe a square $FGHK$ on the side FG of 7.5 cm. [*Prob* 13]. Draw AC, BD the diameters of the square, and let them intersect at E With centre E and radius AE draw the inscribed circle, then it will touch the sides at the pts, BC, D and E,

(ii) Upon folding the square about AC, the st. line BE will fall upon ED, because the $\angle$s AEB, AED are rt. angles, and since $BE=ED$, therefore the pt B will coincide with the pt. D.

$\therefore$ The pts. B and D are symmetrically opposite with regard to AC.

Similarly it may be proved that the pts. A and C are symmetrically opposite with regard to BD.

And since $AC=BD$, therefore the pts. A, B, C, and D lie on the inscribed circle.

4 See fig. Ex, 1.

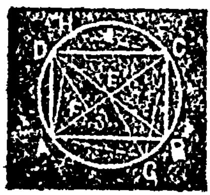
Take a st. line $BC=6$ cm., and describe a square upon it [*Prob.* 13]. Draw diagonals AC, BD intersecting at E. Then AC, and CD are equal and bisect one another at E.

$\therefore$ If a circle be drawn with centre E, passing through any one of the points A, B, C, and D then it will pass through the other three. Draw the circle, then it will circumscribe the square $ABCD$.

Measure the diameter AC, and you will find it= 8 5 cm.

By *calculation* $AC=\sqrt{AB^2+BC^2}=\sqrt{36+36}=8.5$ cm.

5. (i) Take a st. line $AC=3.6''$, and upon R Γ $\frac{1}{6}$



AC as diameter describe a circle. With centres A and C and radii $=3''$ cut the circle at the pts. B and D. Join AD, DC, AB and BC. ***Then ABCD is the required inscribed rectangle.***

The other side $AD=\sqrt{AC^2-CD^2}=\sqrt{12.96-9}=1.99''=$ $2''$ nearly.

(ii) Bisect AC at E, and through E draw the diameter GEH perp to AC. Join AG, CG, AH, and CH. Then $AGCH$ is a square inscribed in the circle $ABCD$. From D draw DF perp. to AC.

Area of the sq. $AGCH=2$ the $\triangle$ $AHC=HE\ AC$.

And area of the rect $ABCD = 2$ the $\triangle ADC = DF \cdot AC$

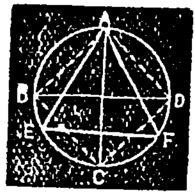
But HE is greater than DF, therefore $HE \cdot AC$ is greater than $DF \cdot AC$.

∴ The area of the sq $AGCH$ is greater than that of the rect. $ABCD$

Similarly it may be proved that it is greater than the area of any other rectangle inscribed in the circle $ABCD$

Of all the rectangles inscribed in a circle, the square has the greatest area

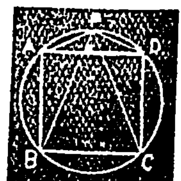
6 Let $ABCD$ be the inscribed square and AEF the inscribed equilateral triangle in the given circle, and let a and b denote their respective sides *It is required to prove that* $3a^2 = 2b^2$



If r denotes the radius of the given circle, then $b = r\sqrt{3}$ [*Ex* 4, P 198], and $a = r\sqrt{2}$ [*Ex* 1]

∴ $b^2 = 3r^2$ and $a^2 = 2r^2$ and therefore $3a^2 = 6r^2 = 2b^2$

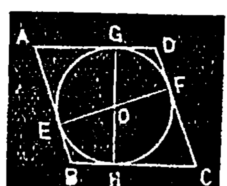
7. Let $ABCD$ be a square inscribed in a given circle, and let P be any point on the arc AD Join AP, BP, CP and DP *Then the* $\angle APD$ *shall be three times any one of the angles* APB, BPC *and* CPD



Because the chords AB, BC, CD are equal to one another, therefore the arcs AB, BC, and CD are also equal [*Th* 44], and therefore the $\angle$s APB, BPC and CPD which stand on these arcs are also equal [*Th* 48]

∴ The $\angle APD = 3$ times any one of the $\angle$s APB, BPC, and CPD

8 Let O be the centre of a given circle *It is required to circumscribe a rhombus about it*



Cons. Draw any two diameters EF and GH intersecting at O At the points E, G, F, and H draw tangents to the

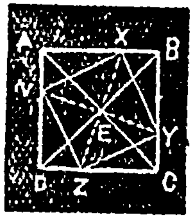
circle meeting one another at the points A, B, C, and D. Then $ABCD$ is the required rhombus

Proof. Because the angles at G and H are rt angles, therefore AD is parl. to BC [*Th.* 13] ¶ Similarly it may be proved that AB is parl. to DC ¶ $\therefore ABCD$ is a parallelogram so that $AD=BC$ and $AB=DC$

But $AD+BC=AB+DC$ [*Ex* 14, *P.* 177] therefore 2 AD =2 AB and therefore $AD=AB$ $\therefore$ The sides AB, BC, CD, and DA are all equal.

$\therefore$ $ABCD$ is a rhombus.

9. Let $ABCD$ be a given square and X a point in one of its sides AB. *It is required to inscribe a square in the sq. $ABCD$, so that one of its angular point may be at X*



Cons. Draw the diagonals AC, BD intersecting at E Join XE, and produce it to meet CD at Z. Through E draw YEV perp to XZ meeting BC, AD at the pts Y and V. Join XY and ZV, then $XYZV$ is the required square

Proof Since $\angle AEB=\angle XEV$ [each being a rt. angle] therefore $\angle AEV=\angle BEX$.

But $\angle AEV=\angle CEY$, and $\angle BEX=\angle DEZ$.

$\therefore$ $\angle CEY=\angle AEV=\angle BEX=\angle DEZ$.

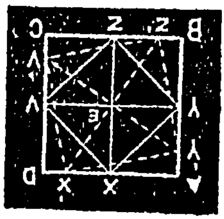
Also $\angle ECY=\angle EAV=\angle EBX=\angle EDZ$ (each being=45°), and the side $EC=EA=EB=ED$.

$\therefore$ The $\triangle$s ECY, EAV, EBX and EDZ are congruent (*Th.17*), so that $EY=EV=EX=EZ$

Again since XEZ and YEV cut at rt. angles at E.

$\therefore$ The fig $XYZV$ is a square.

10 Let $ABCD$ be a given square. *It is required to inscribe in it a square of minimum area.*



Cons. Bisect the sides AB, BC, CD, and DA at the pts Y, Z, V and X respectively Join XY, YZ, ZV and VX. Then $XYZV$ is the required square

Proof. Let $X'Y'Z'V'$ be any other inscribed square Then the diagonals XZ, YV and $X'Z', Y''V'$ will intersect at the same point E.

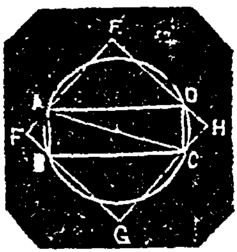
Because EX' is greater than EX, and EY' greater than EY (*Th* 12) therefore $X'Z'$ is greater than XZ, and $Y'V'$ greater than YV.

Again because the area of the sq $XYZV = \frac{1}{2}\, XZ.\ YV$, and the area of the sq. $X'Y'V'Z' = \frac{1}{2}\, X'Z'.\ Y'V'$ (*Ex*. 8, *P*. 113)

. The area of the sq $XYZV$ is less than that of the sq. $X'Y'Z'V'$.

Similarly it may be proved that the area of the sq $XYZV$ is less than that of any other square inscribed in the given square $ABCD$.

$\therefore$ $XYZV$ is the square of minimum area inscribed in the given square $ABCD$.



(i) Join AC. Then because ABC and ADC are rt angled triangles, and AC is their common hypotenuse hence the circle described on the diameter AC passes through the pts B and D, and is therefore the circumscribed circle of the rectangle $ABCD$

(ii) At the pts A and D make the $\angle$s DAE, ADE each $=$ 45°, then the $\angle AED = 90°$ Through the pts C and B draw st lines parallel to AE DE respectively meeting EA, ED produced at the pts F and H. *Then the figure EFGH thus formed will be the required square*

Fig $EFGH$ is by construction a parallelogram and because the $\angle$ AED is a rt angle hence it is a rectangle.

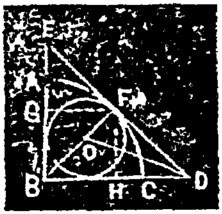
Because FAB and DCH are rt. angled isosceles triangles and have their bases AB and DC equal, therefore they are identically equal. [*Th* 17].

$\therefore FA=DH$. Also $AE=ED$. Therefore $EF=EH$.

$\therefore EF=EH=FG=GH$, and hence the rectangle $EFGH$ is equilateral.

$\therefore$ The figure $EFGH$ is a square.

12. Let ABC be a given quadrant. *It is required to inscribe (i) a circle and (ii) a square in the quadrant ABC.*



(i) Bisect the angle ABC, by the st. line BF meeting the arc AC at F, and at the point F draw a tangent to meet BC, BA produced at the points D and E respectively.

Bisect the $\angle BDE$ by the st. line DO meeting BF at O. Then O is the in-centre of the triangle BDE [*Prob.* 26].

The circle inscribed in the triangle BDE is the required circle, because it touches each of the sides BA and BC; and since it touches the tangent DE at F it also touches the arc AC at F.

(ii) From F draw FG, FH perps. to BA, BC respectively. Then the fig. $GBHF$ is by construction a rectangle.

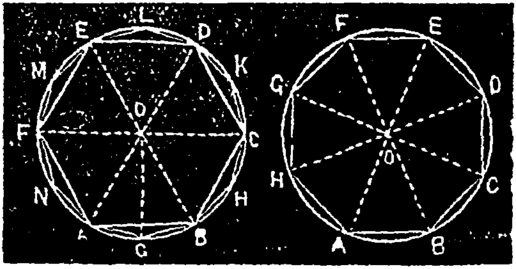
Because the $\angle FBH=45°$ [*Cons*] and the $\angle FHB=90°$, therefore the $\angle BFH=45°$ ¶ $\therefore$ The $\angle FBH=$ the $\angle BFH$, and therefore $BH=FH$.

$\therefore BH=HF=BG=GF$, and hence the rectangle $FGBH$ is equilateral.

$\therefore$ The fig. $FGBH$ is a square, and it is inscribed in the quadrant ABC.

Exercises on Problem 30, P. 200.

FIG (i) FIG (ii)



1. (i) With any point O as centre and radius=4 cm draw a circle, and let AOD be one of its diameters With centre A and radius OA draw an arc cutting the circumference at B and F. Through B and F draw the diameters BOE and FOC, then each of the angles at O is evidently$=\dfrac{360°}{6}=60°$.

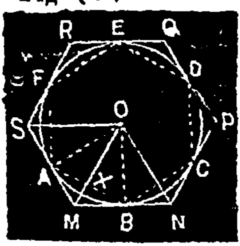
Join AB, BC, CD, DE. EF and FA, then $A\,B\,C\,D\,E\,F$ is the required regular hexagon (*Prob.* 30)

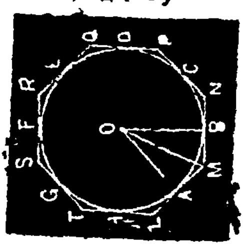
(ii) Draw any two diameters AOE, COG intersecting at right angles at the centre O Draw two other diameters BOF, HOD bisecting the angles between the first two diameters. then each of the angles at O is evidently$=\dfrac{360°}{8}=45°$

Join AB, BC, CD, DE, EF, FG GH and HA then $ABCDEFGH$ is the required regular octagon [*Prob* 30].

(iii) Bisect the arcs AB, BC, CD, DE, EF and FA in fig. (i) at the pts G, H, K, L, M and N; and join AG, GB, BH, HC CK, KD, DL, LE, EM, MF FN and NA. Then the resulting figure $A\,G\,B\,H\,C\,K\,D\,L\,E\,M\,F\,N$ is the required regular do-decagon.

Fig (i) Fig.(ii)





2. (i) With any point O as centre, and radius=1 5" draw a circle. Determine A, B, C,D, E, and F the angular points of a egular hexagon inscribed in this circle

(as in *Ex.* 1. (i)], and draw tangents to the circle at these points. *The resulting figure MNPQRS shall be the circumscribed regular hexagon.*

Proof. Join OM, ON and OS. Then because the angles at A and B are rt. angles, therefore the $\angle$s AOB, AMB together$=$2 rt. angles [*Th.* 16. *Inf.* 5].

But the $\angle$ $AOB=60°$, $\therefore$ therefore the $\angle$ $AMB=120°$.

Similarly it may be proved that each of the $\angle$s at N, P, Q, R and $S=120°$ $\therefore$ The figure is equiangular.

Again because the circle touches the st. lines MS and MN, therefore OM bisects the $\angle$ SMN.

Similarly ON, OS bisect the $\angle$s at N and S respectively

$\therefore$ The $\angle$s OSM, OMS, OMN and ONM are each$=60°$.

$\therefore$ The equilateral $\triangle$s OMS and OMN are identically equal (*Th.* 17), and therefore $MN=MS$.

Similarly the other sides of this figure are equal $\therefore$ The figure is equilateral.

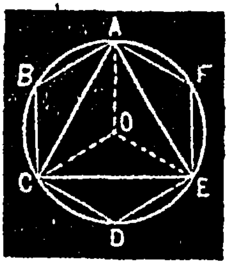
But it has been proved to be equiangular, therefore it is regular.

Verify by *measurement* that the sides of this figure are equal and each of its angles$=120°$.

Draw a circle as in case (i) Determine A, B, C, D, E, F, G and H, the angular points of a regular octagon inscribed in this circle *as in Ex* 1, *case* (iii)]. and draw tangents to the circle at these points *The resulting figure LMNPQRST shall be the circumscribed regular octagon.*

For proof proceed as in case (i).

3 Let O be the centre of a given circle Inscribed a regular hexagon $ABCDEF$ in its Join AC, CE and EA. Then ACE is the inscribed equilateral triangle in it Let a and b denote the lengths of the sides of the triangle and the hexagon respectively *It is*



required to prove that (I) *the area of the triangle$=\frac{1}{2}$ the area*

of the hexagon, and (ii) $a^2 = 3b^2$.

(i) Join OA, OC and OE Then because AC is the side of an equilateral triangle therefore the $\angle AOC = \frac{360}{3} 120°$.

Also the $\angle ABC$ being the angle of a regular hexagon= 120°.

∴ The opp $\angle$s at B and O are equal, and therefore the fig' $ABCO$ is a parallelogram Similarly it can be shown that $OCDE$ and $OEFA$ are parallelograms

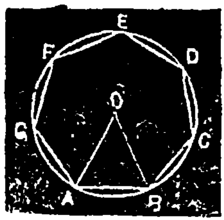
Because the $\triangle$s AOC OCE, and EOA are respectively $= \frac{1}{2}$ the parms $AOCB$, $OCDE$ and $OEFA$.

∴ The $\triangle ACE = \frac{1}{2}$ the hexagon $ABCDEF$

(ii) Because $AC^2 = 3\ OC^2$ [*Ex* 4, *P.* 198], and $OC = AB$

∴ $AC^2 = 3\ AB^2$, that is $a^2 = 3b^2$.

4 With any point O as centre, and radius= 2" draw a circle By means of your protactor make the $\angle AOB = \frac{360°}{7}$ Set off the chords BC. CD, DE, EF, and FG, each equal to AB round the circumference, and join GA *Then* $ABCDEFG$ *is the required heptagon inscribed in the given circle.*



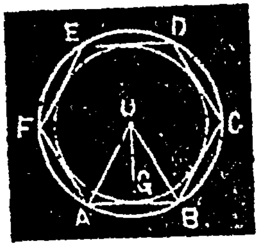
Because 7 times the $\angle ABC + 360° = 14 \times 90°$ (*Th 16,Cor*)

∴ The $\angle ABC = \frac{14 \times 90 - 360}{7} = 128\frac{4°}{7}$

Measure the $\angle ABC$ and the side AB, and you will find them $= 128\frac{4}{7}°$ and 1 73" respectively.

Exercises on Problem 30, P. 201.

1. Take a st. line $AB = 2''$, and make the $\angle s\ BAF, ABC$ each $= 120^\circ$, such that AF, BC each $= 2''$. At the points F and C make the $\angle s\ AFE, BCD$ each $= 120^\circ$ making FE, CD each $= 2''$. Join ED. *Then ABCDEF is the required hexagon on a side of 2''.*



Bisect the $\angle s\ ABC, BAF$ by the st. lines BO, AO meeting at O. With centre O and radius OA describe a circle, then this will be the circumscribed circle of the hexagon $ABCDEF$ [*Prob* 31].

From O draw OG perp. to AB. With centre O, and radius OG describe a circle then this will be the inscribed circle of the hexagon $ABCDEF$ [*Prob* 31].

By *calculation* $OA = AB = 2''$, therefore the circum-diameter $= 4''$.

And $OG = \sqrt{OA^2 - AG^2} = \sqrt{4 - 1} = \sqrt{3} = 1.732''$ therefore the in-diameter $= 3.46''$.

Measure the circum-diameter, and the in-diameter, and you will find them $= 4''$ and $3.46''$ respectively.

2. Let O be the centre of the given circle, and let $ABCDEF$ and $MNPQRS$ be the inscribed and the circums-cribed regular hexagons. *It is required to prove that the area of* $ABCDEF = \frac{3}{4}$ *of the area of MNPQRS.*

Join OA, OB and OM, ON, and let OM cut AB at X.

Then $OX = \sqrt{(OA^2 - AX^2)} = \sqrt{(OA^2 - \frac{1}{4}OA^2)} = \frac{\sqrt{3}}{2}\ OA$, and $OA = \sqrt{(OM^2 - AM^2)} = \sqrt{(OM^2 - \frac{1}{4}OM^2)} =$

$\frac{\sqrt{3}}{2}OM = \frac{\sqrt{3}}{2}MN$. Therefore $MN = \frac{2}{\sqrt{3}}\ OA$.

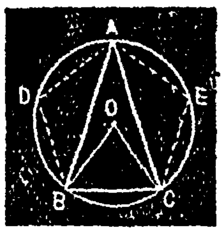
Because the $\triangle OAB = \frac{1}{2}AB\,OX = \frac{1}{2}OA \times \frac{\sqrt{3}}{2}\ OA$ $= \frac{\sqrt{3}}{4}\ OA^2$, and the $\triangle OMN = \frac{1}{2}OA.MN = \frac{1}{2}OA \times \frac{2}{\sqrt{3}}OA = \frac{1}{\sqrt{3}}\ OA^2$.

$\therefore \triangle OAB = \frac{3}{4}\triangle OMN$

Again because the hexagon $ABCDEF = 6\ \triangle OAB$, and the hexagon $MNPQRS = 6\ \triangle OMN$

$\therefore$ The hexagon $ABCDEF = \frac{3}{4}$ of the hexagon $MNPQRS$.

If $OA = 10$ cm, then the area of the hexagon $ABCDEF = 6\ \triangle OAB = 6 \times \frac{\sqrt{3}}{4}\ OA^2 = 6 \times \frac{\sqrt{3}}{4} \times 10^2 =$ $150\sqrt{3} = 259.8$ sq cm.



3 Let O be the centre of a given circle, and let ABC be an isosceles triangle inscribed in it, such that each of the angles B and C is double of the angle A. *Then BC is a side of a regular pentagon inscribed in the circle*

Because the $\angle$s A, B and C together $= 180°$ [*Th* 16], and the $\angle$s B and C are each of them $= 2$ the $\angle A$

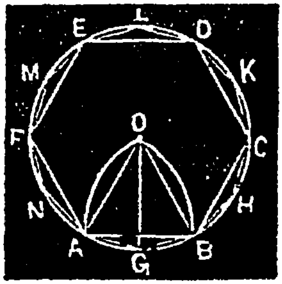
$\therefore$ 5 times the $\angle A = 180°$, and therefore $\angle A = 36°$.

Join OB, OC. Then the $\angle BOC = 2$ the $\angle BAC$ [*Th.* 38] $= 72° = \frac{360°}{5}$.

$\therefore$ BC is a side of a regular pentagon inscribed in the given circle [*Prob* 30].

Note—*See also Ex* 17, *P.* 170-71.

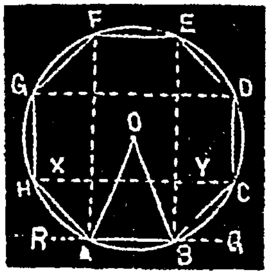
4. Take a st. line $AB=4$ cm. *It is required to construct* [without protactor] (i) *a regular hexagon and* (ii) *a regular octagon on it.*



(i) With centres A and B and radius $=4$ cm draw two arcs intersecting in the point O. With certre O and radius OA describe a circle, and set off round the circumference chords BC, CD, DE and EF each equal to AB. Join FA. Then $ABCDEF$ is the required hexagon.

Area of the hexagon $ABCDEF=6$ times the $\triangle\ OAB$.

$$=6\times\frac{\sqrt{3}}{4}AB^2[Ex.\ 2]=6\times\frac{\sqrt{3}}{4}\times 16=31.57 \text{ sq cm.}$$



(ii) Take a st line $AB=4$ cm. Produce it both ways to any pts R and Q and draw AF, BE perps. to AB. Bisect the $\angle$s FAR and EBQ by AH, BC making AH, BC each$=4$ cm Draw HG CD parl to AF, BE and make HG, CD each$=4$ cm With centres G and D and radius$=4$ cm draw two arcs cutting the lines AF and BE at the pts. F and E. Join GF, DE and EF. *Then $ABCDEFGH$ is the required octagon.*

Join GD and HC, then the figure is divided into 4 rt. angled isosceles triangles, four rectangles and a central square.

Let HC cut AF at X, and BE at Y. Then $AH^2=AX^2+HX^2=2AX^2$. Therefore $AX=\frac{AH}{\sqrt{2}}=\frac{4}{\sqrt{2}}=2\sqrt{2}$ cm.

$\therefore$ Area of th figure $=4\triangle AHX+4$ rect. $AB, AX+XY^2$
$=4(\frac{1}{2}\ AX\ HX)+4(AB\ AX)+AB^2$
$=4\times(\frac{1}{2}\times 2\sqrt{2}\times 2\sqrt{2})+4\times(4\times 2\sqrt{2})$
$+4^2$
$=16+32\sqrt{2}+16=77.25$ sq cm

Exercises on the circumferences of a circle. Page 202.

1. Because $\pi=\dfrac{\text{circumference}}{\text{diameter}}$

$\therefore$ In case (i). $\pi=\dfrac{16}{5.1}=3\ 13725$ cm.

In case (ii) $\pi=\dfrac{8\ 8}{2.8}=3\ 14286$ cm.

And in case (iii) $\pi=\dfrac{13\ 5}{4\ 3}=3\ 14186$ cm.

$\therefore$ Mean of the three results $=\dfrac{3\ 13725+3\ 14286+3\ 14186}{3}$
$=3.14065.$

2. Length required for 20 complete turns $=75\ 4''$

$\therefore$ ... . 1 .... turn $=3\ 77''$.

$\therefore\ \pi=\dfrac{\text{circumference}}{\text{diameter}}=\dfrac{3\ 77}{1\ 2}$ 3.1417" nearly.

3. Because the wheel makes 400 revolutions in 977 yds.

$\therefore$ . ...1 revolution in 2 4425 yds.

$\therefore\ \pi=\dfrac{\text{circumference}}{\text{diameter}}=\dfrac{2.4425\text{ yds.}}{28''}=\dfrac{2.4425\times 3\times 12}{28}$
$=3\ 140357$

Exercises on the area of a circle, P. 205.

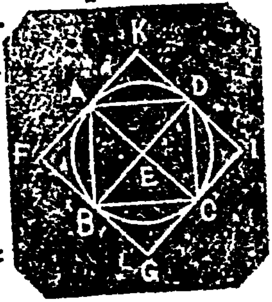
1. (i) circumference of a circle of 4 5 cm, radius $=2\pi r$
$=2\times 3.1416\times 4.5=28\ 3$ cm.

(ii)................................100 cm............$=2\pi r$
$=2\times 3.1416\times 100=6283$ cm.

2 (i) Area of a circle of 2.3" radius $= \pi r^2 = 3.1416 \times (2.3)^2$
$= 16.62$ sq. In.

(ii) ....10.6"... $= \pi r^2 = 3.1416 \times (10.6)^2$
$= 352.99$ sq. In.

3. AE the radius of a circle inscribed in a square of 3.6 cm side $= \frac{1}{2} AG = \frac{1}{2} FG = 1.8$ cm.



$\therefore$ Circumference $= 2\pi r = 2 \times 3.1416 \times 1.8 = 11.3$ cm.,

and $area = \pi r^2 = 3.1416 \times (1.8)^2 = 10.18$ sq. cm.

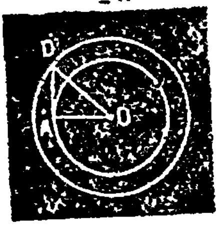
4. See fig. Ex. 3.

Diagonal AC of the square inscribed in a circle of 7 cm. radius $= 14$ cm. ¶ $\therefore$ Its area $= \frac{1}{2} \times 14 \times 14 = 98$ sq. cm.

Area of the circle $= \pi r^2 = \frac{22}{7} \times (7)^2 = 154$ sq. cm

$\therefore$ Difference of the areas $= 154 - 98 = 56$ sq. cm.

R. F. $\frac{3}{2\pi}$



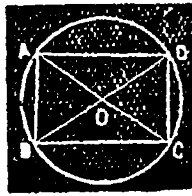
5. Let O be the common centre of two concentric circles of radii 5.7" and 4.3".

Then the area of the circular ring thus formed $= \pi OD^2 - \pi OA^2 = \pi [(5.7)^2 - (4.3)^2]$.
$= 3.1416 \times 14 = 43.98$ sq In.

6. See fig. Ex. 5

Let O be the centre of two concentric circles, and let DA be drawn tangent to the inner circle from any pt D on the outer circle.

Area of a circle of radius $AD = \pi AD^2$.
$= \pi (OD^2 - OA^2)$.
$=$ area of the ring.
[*Ex.* 5].



7 Diameter of the circum circle of a rectangle $ABCD$ of sides 8 cm and 6 cm, = the diagonal of the rectangle $ABCD$

But the diagonal $AC = \sqrt{AD^2+DC^2}$ $= \sqrt{(8^2+6^2)} = 10$ cm, ∴ The circumradius $OA = 5$ cm.

Area of four segments outside the rectangle
= difference of the areas of the circle and the rectangle

$= \pi OA^2 - AB\cdot AD = 3\cdot1416 \times 5^2 - 8 \times 6 = 30\cdot5$ sq. cm.

8 Because the area of the circle of radius $5'' = \pi \times (5)^2$

∴ Area of the required square $= \pi \times (5)^2 = 78\cdot54$ sq. In.

∴ Sides of the required square $= \sqrt{(78\cdot54)} = 8\cdot9''$ *nearly*.

9 See fig. Ex 5.

Let x'' be the radius of the smaller of the two concentric circles Then since the width of the ring $= 1''$

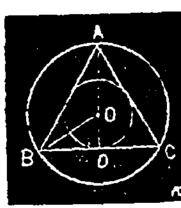
∴ The radius of the greater circle $= (x+1)''$.

∴ Area of the ring $= \pi \left\{ (x+1)^2 - x^2 \right\} = \dfrac{22}{7}(2x+1)$ sq In.

But the area of the given ring $= 22$ sq In.

∴ $\dfrac{22}{7}(2x+1) = 22$, and therefore $x = 3$.

∴ The radii are $4''$ and $3''$



10. Draw the equilateral triangle ABC whose sides $= 4''$ Draw the circumscribed and the inscribed circles as shown in the figure Then these circles are concentric Let O be the common centre.

Then because $AD = \sqrt{AB^2 - BD^2} = \sqrt{16-4} = 2\sqrt{3}$.

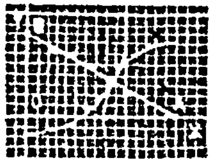
∴ $AO = \frac{2}{3}AD = \frac{2}{3} \times 2\sqrt{3} = \dfrac{4\sqrt{3}}{3}$, and $OD = \frac{1}{2}AO = \dfrac{2\sqrt{3}}{3}$

[Ex. 2 P. 198].

∴ Difference of the areas of these two circles.

$=\pi(AO^2-OD^2)=3.1416\times(\frac{16}{3}-\frac{4}{3})=12.57$ sq In.

11 Let A and B be the points (1.5",0) and (0,.8") respectively. Join AB. Then AB $=\sqrt{OA^2+OB^2}=\sqrt{(.8)^2+(1.5)^2}=1.7''$.

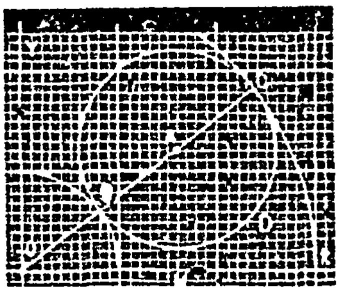


With centres A and B and radii .7" and 1.0" draw two circles, then they will touch each other externally, because the sum of their radii $=.7+1.0=1.7''=$ the distance between their centres A and B [*Cor.* 1, *P.* 178]

∴ Their circumferences $=2\times3.1416\times.7=4.4''$, and $2\times3.1416\times1=6.3''$ approximately.

Their areas $=3.1416\times(.7)^2=1.54$ sq. In. and $3.1416\times(1)^2=3.1416$ sq In. approximately

12 Let A be the point (1.6",1.2")



With centre A and radius 1" describe the circle BCD. Join OA, and let OA cut the circle BCD at the points B and C.

Then $OA=\sqrt{OM^2+AM^2}=\sqrt{(1.6)^2+(1.2)^2}=2''$.

$\therefore OB=OA-OB=2-1=1''$, and $OC=OA$ and $OC=2+1=3''$.

∴ The circles described with centre A and radii 1" and 3" will touch the circle BCD, the former *externally* at B, and the latter *internally* at C respectively. Draw the circles as shown in the figure

Exercises on the Inscribed and the Escribed circles. P. 206.

1. Let AB and CD be any two parallel straight lines and EF any other straight line meeting them. *It is required to describe circles to touch AB, CD and EF.*

(i) Locus of the centres of circles touching AB and EF is one or other of the st. lines EG, EH bisecting the angles AEF and BEF respectively. [*Note VI, P* 188]

(ii) Locus of the centres of circles touching CD and EF is one or other of the st. lines FG, FH bisecting the angles CFE and DFE respectively [*Note VI, P* 188].

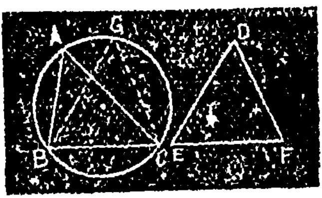
∴ The pts. G and H where these st. lines intersect are the centres of the required circles.

(iii) Because the centres of all circles touching two parallel st. lines lie on a st. line parallel to the given lines and midway between them [*Note VI, P*, 188]

∴ The pts G and H are equally distant from CD, hence the radii of the two circles are equal

∴ The two circles are also equal

2. Let ABC, DEF be two triangles which have their bases $BC = EF$, and the vert $\angle BAC =$ the vert $\angle EDF$, *then their circum-circles shall be equal*



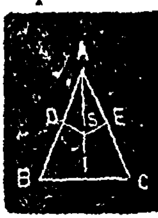
Place the triangle DEF over the triangle ABC such that the pt E falls on B and EF along BC then because $EF = BC$ therefore EF will coincide with BC.

Let BGC represent the new position of the $\triangle EDF$, then because the $\angle BGC =$ the $\angle BAC$ therefore the pts A, G, C and B are concyclic [*Converse of Th. 39*]

∴ The circum-circle of the $\triangle ABC$ is also the circum-circle of the $\triangle BGC$.

∴ The circum-circles of the $\triangle$s ABC and DEF are equal.

3. Let ABC be a triangle, and let I be its in-centre. Join AI, and let the circumcentre S lie on AI. *Then shall AB be equal to AC.*



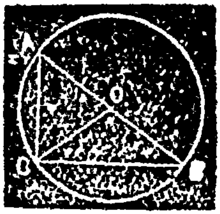
Because I is the in-centre, therefore the $\angle BAI =$ the $\angle CAI$ [*Prob.* 26].

From S draw SD, SE perpendiculars on

AB and AC respectively. Then since S is the circum centre therefore D and E are the middle points of AB and AC respectively [*Prob.* 25].

In the triangles SAD and SAE, because the rt. $\angle SDA$ = the rt. $\angle SAD$, the $\angle SAD$ = the $\angle SAE$, and the side AS is common to both therefore the triangles are identically equal [*Th.* 17].

$\therefore AD = AE$, and therefore also $AB = AC$.



4. Let ABC be a triangle rt. angled at C, and let d, D denote the diameters of its inscribed and circumscribed circles. *Then shall* $D+d=a+b$.

Because the area of the $\triangle ABC = \frac{1}{2}(a+b+c)\times r$ [*Ex.* 6, P. 198] and it is also $=\frac{1}{2}ab$.

$\therefore \frac{1}{2}ab=\frac{1}{2}(a+b+c)\times r$, whence $r=\dfrac{ab}{a+b+c}$, and therefore

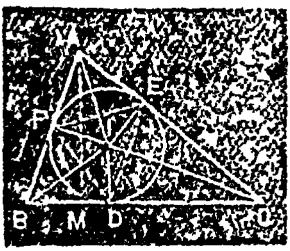
$$d=\frac{2ab}{a+b+c}$$

Again because the $\angle C$ is a rt angle therefore $c^2=a^2+b^2$, and $D=AB=c$ [*Prob* 10].

$$\therefore D+d=c+\frac{2ab}{a+b+c}=\frac{c(a+b)+c^2+2ab}{a+b+c}$$

$$=\frac{c(a+b)+(a^2+b^2)+2ab}{(a+b+c)}=\frac{c(a+b)+(a+b)^2}{(a+b+c)}$$

$$=\frac{(a+b)(c+a+b)}{(a+b+c)}=a+b$$



5. Let ABC be a triangle, and let the inscribed circle touch the sides AB, BC and CA at the pts. F, D and E respectively *Then the angles of the triangle DEF shall be respectively* $90°-\frac{A}{2}$, $90°-\frac{B}{2}$, $90°-\frac{C}{2}$

Because AE and AF are two tan-

gents drawn from A, therefore $AE=AF$ (*Th* 47), & therefore the $\angle AFE=$ the $\angle AEF$.

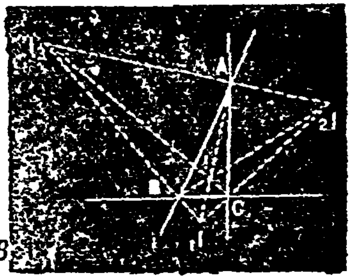
Again because the $\angle AFE+$ the $\angle AEF+$ the $\angle FAE=180°$ that is 2 the $\angle AFE+$ the $\angle A=180°$

$\therefore$ The $\angle AFE+\frac{a}{2}=90°$, and therefore the $\angle AFE=90°-\frac{a}{2}$

But the $\angle AFE=$ the $\angle FDE$ in the alt. segment [*Th* 49]. Therefore the $\angle FDE=90°-\frac{a}{2}$

Similarly it can be proved that the $\angle FED=90°-\frac{b}{2}$, and the $\angle EFD=90°-\frac{c}{2}$.

6 Let ABC be a triangle, and let I, I_1 be the centres of the inscribed and the escribed circles *Then I, B, I_1 and C shall be concyclic.*



Because IB, IC bisect the int $\angle s$ B and C [*Prob* 26], and I_1B, I_1C bisect the ext $\angle s B$ and C [*Prob* 27]

$\therefore$ The $\angle s$ IBI_1 and ICI_1 are rt angles.

$\therefore$ The pts I, B, I_1 and C are concyclic [*Converse of Th.* 40].

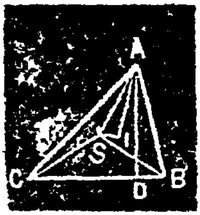
7. See fig Ex 5.

Let ABC be a triangle, and F, D, E the pts. where the in-circle touches the sides AB, BC, and CA respectively. *Then shall $AC-AB=CD-BD$*

Because $AE=AF$, $BD=BF$, and $CE=CD$ [*Th*, 47]

$\therefore$ $AC-AB=(EA+CE)-(AF+BF)=(AF+CD)-(AF+BD)$ $=CD-BD$.

8. (i) Let ABC be a triangle, and let I and S be the centres of the inscribed and the circumscribed circles. Join IS, AI and AS. *Then shall the* $\angle IAS = \frac{1}{2}(B-C)$.



Join SB, SC. Then because $SB = SC$ (*each being circumradius*), therefore the $\angle SBC =$ the $\angle SCB$.

Similarly the $\angle SBA =$ the $\angle SAB$, and the $\angle SCA =$ the $\angle SAC$.

$\therefore B - C =$ (the $\angle ABS +$ the $\angle CBS$) - (the $\angle ACS +$ the $\angle CBS$)

$=$ the $\angle ABS -$ the $\angle ACS$.

$=$ the $\angle BAS -$ the $\angle CAS$.

$=$ (the $\angle BAI +$ the $\angle IAS$) - (the $\angle CAI -$ the $\angle IAS$)

$= 2$ the $\angle IAS$ [$\therefore$ the $\angle BAI =$ the $\angle CAI$]

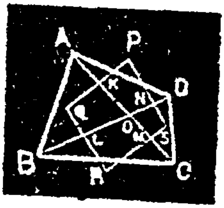
.. The $\angle IAS = \frac{1}{2}(B-C)$

(ii) From A draw AD perpendicular to BC. Then since IA is the bisector of the vertical $\angle BAC$, therefore the $\angle DAI = \frac{1}{2}(B-C)$ [*Ex.* 3, *P* 138].

$\therefore$ The $\angle DAI =$ the $\angle IAS$.

$\therefore$ AI is the bisector of the angle DAS.

9. Let $ABCD$ be a quadrilateral of which the diagonals AC, BD intersect at O. Bisect AO, BO, CO and DO at the points K, L, M, N respectively. Let the perpendiculars at K, L, M and N to the straight lines AO, BO, CO and DO meet at the points P, Q, R and S as shown in the figure Then P, Q, R and S are the circum-centres of the triangles AOD AOB, BOC and COD respectively [*Prob.*25].



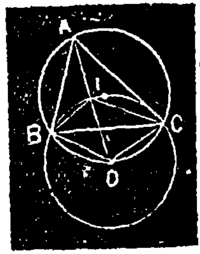
It is required to prove that PQRS is a parallelogram.

Because PQ, RS are each of them perp. to AC.

$\therefore PQ$ is parallel to RS [*Ex* 2 *P*. 41]

Again because PS, QR are each of them perp to BD therefore PS is parallel to QR [*Ex* 2 *P* 41]

$\therefore PQRS$ is a parallelogram.



10 Let ABC be a triangle, and let I be its in-centre Describe a circle about it [*Prob*. 25], and produce AI to meet this circle at O Then *shall O be the centre of the circle circumscribed about the triangle BIC*

Join BI, CI, BO and CO Then because I is the in-centre therefore AI, BI and CI bisect the $\angle$s A, B, and C respectively [*Prob*. 26]

$\therefore$ The $\angle OIB$=the $\angle IAB$+the $\angle IBA$ [*Th*. 16]= $\frac{a}{2}+\frac{b}{2}$

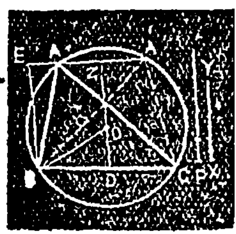
Again because the $\angle OBC$=the $\angle OAC$ [*Th* 39] $=\frac{a}{2}$ and the $\angle CBI=\frac{b}{2}$

$\therefore$ the $\angle OBI$=the $\angle OBC$+the $CBI=\frac{a}{2}+\frac{b}{2}$

$\therefore$ The $\angle OBI$=the $\angle OIB$, and therefore $OB=OI$

Similarly it may be proved that $OC=OI$ ¶ $\therefore OB=OI=OC$.

$\therefore O$ is the centre of the circle described about the $\triangle$ BIC [*Th* 33].



11 Let BC be the base, P the altitude, and XY the radius of the circumscribed circle of a triangle *It is required to construct it*

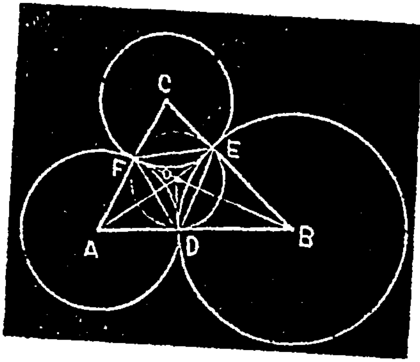
Bisect BC at D, and draw DZ perp to BC Then the circum-centre lies on DZ [*Prob*. 25]. With centre B and radius=XY draw an arc cut-

ting DZ at the pt. U. Then O is the circum-centre of the required triangle.

With centre O and radius OB describe the circle BAC.

Draw BE perpendicular to BC and make it equal to P. From E draw the st line EAF parl. to BC, cutting the circle at the pt A and A'. Join AB, AC and $A'B$, $A'C$.

Then ABC and $A'BC$ are the required triangles



12 Let A, B and C be the centre of three circles touching one another externally, two by two, at the pts D E and F. *It is required to prove that the inscribed circle of the triangle ABC is the circumscribed circle of the triangle DEF.*

Bisect the $\angle$s A and B by the st. lines AO and BO intersecting at pt. O, then O is the centre of the inscribed circle of the $\triangle$ ABC. [*Prob.* 26].

Join OD, OE and OF. Then in the $\triangle$s AOD and AOF, because the side $AD = AF$ and the side AO is common to both, also the $\angle OAD$ = the $\angle OAF$; therefore the triangles are identically equal [*Th.* 4], and therefore $OD = OF$.

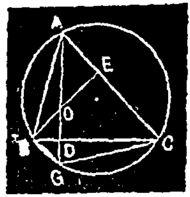
Similarly it can be proved that $OD = OE$.

$\therefore$ A circle drawn with centre O and radius OD, must also pass through the pts E and F, and is therefore the circumscribed circle of the $\triangle$ DEF.

Again because O is the in-centre of the $\triangle$ ABC, and OD, OE, OF are perps. to AB, BC and CA respectively hence the circle drawn with centre O and radius OD is the inscribed circle of the $\triangle ABC$.

$\therefore$ The inscribed circle of the $\triangle$ ABC is the circumscribed circle of the $\triangle$ DEF.

Exercises on the Orthocentre, P. 209



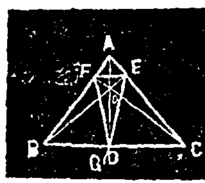
1. Let O be the orthocentre of the $\triangle ABC$, and let the perpendicular AD produced meet the circum-circle at the pt G. *Then shall* $OD = DG$.

Join BO, BG and CG, and produce BO to meet AC at the pt E.

Then the $\angle GBE =$ the $\angle CAD$ (*For each* $= 90° - \angle ACD$)
$=$ the $\angle CBG$ in the same segment.

Now in the $\triangle$s OBD and GBD, because the $\angle OBD =$ the $\angle GBD$, (*Proved*) and the rt. $\angle ODB =$ the rt $\angle GDB$, and the side BD is common to both, therefore the triangle are identically equal [*Th.* 17].

$\therefore OD = DG$



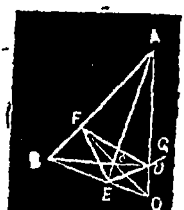
2 (i) Let DEF be the pedal triangle of the acute angled $\triangle ABC$. Produce ED to any point G.

Then the $\angle BDF =$ the $\angle EDC$ (*Cor.* 1, P. 208).

$=$ opp $\angle BDG$

$\therefore BD$, or BC bisects the ext. $\angle FDG$, and is therefore, the *external* bisector of the $\angle EDF$.

Similarly it can be proved that CA and AB are the *external* bisectors of the $\angle$s DEF and EFD respectively



(ii) Let DEF be the pedal triangle of the $\triangle ABC$, obtuse-angled at C. Produce ED to any pt G, then AD bisects the ext. $\angle FDG$ [*Note on Cor. 1, P. 208*]

$\therefore$ the $\angle FDA =$ the $\angle ADG =$ the $\angle EDO$.

But the $\angle FDA$s the complement of $\angle BDF$, and the $\angle EDO$ is the complement of the $\angle BDE$, therefore the $\angle BDF =$ the $\angle BDE$.

∴ BC or BD is the *internal*, bisector of the $\angle EDF$.

Similarly it can be proved that AC, or AE is the *internal* bisector of the $\angle DEF$.

3 See fig Ex. 2. (i)

Let O be the orthocentre of the triangle ABC. *It is required to prove that the $\angle$s BOC, BAC are supplementary.*

Because each of the $\angle$s CAD and EBC is the complement of the $\angle$ ACB, therefore the $\angle$ CAD = the $\angle EBC$.

Again because each of the $\angle$s FCB and BAD is the complement of the $\angle ABC$, therefore the $\angle FCB$ = the $\angle BAD$.

∴ the $\angle EBC$ + the $\angle FCB$ = the $\angle CAD$ + the $\angle BAD$.
= the $\angle BAC$.

But the $\angle$s EBC and FCB together are supplement of the $\angle BOC$ [*Th.* 16], therefore the $\angle$s BOC and BAC are supplementary.

4. See fig. Ex. 2 (i)

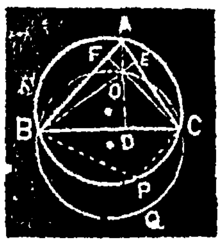
It is required to prove that each of the four points O, A, B and C is the orthocentre of the triangle whose vertices are the other three.

Point O is evidently the orthocentre of the $\triangle ABC$.

In the $\triangle OBC$, because OD, BF and CE are perpendiculars drawn from the vertices to the opp. sides, and because these prependiculars meet at A, therefore the pt. A is the orthocentre of the $\triangle OBC$.

Similarly it can be proved that B is the orthocentre of the $\triangle OAC$, and C is that of the $\triangle OAB$.

5. Let O be the orthocentre of the $\triangle ABC$. *It is required to prove that the circles circumscribed about the triangles OBC, OCA and OAB are each equal to the circum circle of the $\triangle ABC$.*



Join BO, OC and circumscribe the circle $BACP$ and $BOCQ$ about the triangles

BAC and *BOC* respectively [*Th* 25]. Join *BP* and *CP*

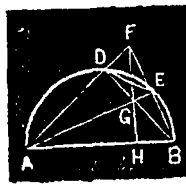
Then the ∠s *BAC* and *BPC* are supplementary [*Th.* 40].

Also the ∠s *BAC* and *BOC* are supplementary [*Ex* 3].

The ∠ *BPC* = the ∠ *BOC*, and therefore the segment *BOC* = the segment *BPC*, and the segment *BAC* = the segment *BQC*

∴ The circle *BACP* = the circle *BOCQ*

Similarly it can be proved that the circle circumscribed about the triangles *OAB* and *OAC* are each equal to the circle *BACP*

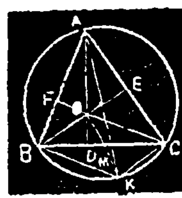


6. Let *ADEB* be a semi-circle described on the diameter *AB*, and let *AD* and *BE* produced intersect at *F*, and *AE* *BD* at *G* Join *FG*, and produce it to meet *AB* at *H*. *Then GH shall be perpendicular to AB*

Each of the ∠s *ADB*, *AEB* being an angle in a semi-circle is a rt angle (*Th* 41), therefore *BD*, *AE* are perpendiculars drawn from the vertices to the opposite sides of the △ *AFB*.

∴ The pt. *G* where *BD* and *AE* intersect is the orthocentre of △ *AFB*

FGH is perpendicular to *AB*



7. Let *ABC* be a triangle, and *O* its orthocentre Describe a circle about the triangle *ABC* (*Prob.* 25) and let *AK* be its diameter. *Then shall the fig BOCK be a parallelogram*

Because the ∠s *BAC* and *BKC* are supplementary (*Th* 40,) as also the angles *BAC* and *BOC* are supplementary (*Ex.* 3), therefore the ∠ *BOC* = the ∠ *BKC*.

Because each of the ∠s *ABK* and *ACK* is a right angle (*Th* 41), therefore they are equal. Also the ∠ *ABE* =

the $\angle ACF$ (*Because each of them is the complement of the $\angle$ BAE*). Therefore the remaining $\angle$s EBK and FCK are also equal.

$\therefore$ The opposite angles of the figure $BOCK$ are equal

$\therefore$ $BOCK$ is a parallelogram (*Ex.* 2, P. 59)

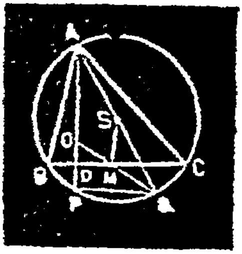
8. See fig Ex. 7.

Join OK. *Then OK shall pass through the middle point of BC.*

Let OK cut BC at M. Then since $BOCK$ is a parallelogram and BC, OK are its diagonals.

$\therefore$ BC and OK bisect each other at M (*Cor* 3. *P.* 59).

$\therefore$ OK passes through the middle point of BC



9 Let ABC be a triangle, and O it orthocentre. Bisect BC at M. Join OM, and produce it to meet the circum-circle at Q Let the prependicular AD produced meet this circle at P. Join PQ *Then PQ shall be parallel to BC.*

In the triangle OPQ because $OD = OP$ (*Ex.* 1) and $OM = MQ$ [*Ex* 8].

$\therefore$ DM or BC is parallel to PQ.

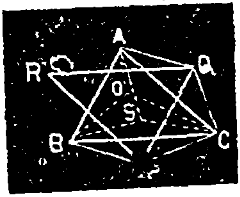
10 See fig Ex. 9.

Let O be the orthocentre, and S the circum-centre of the triangle ABC. *It is required to prove that the distance of each vertex from O is double of the perp. drawn from S on the opposite side*

Join AS, and produce it to meet the circle at the point Q. Join AO and OQ, then OQ passes through the middle point M of BC (*Ex* 8).

Join SM then SM is perp. to BC (*Th.* 31.)

Because $AS = SQ$, and $OM = MQ$, therefore $SM = \frac{1}{2}\, OA$ [*Ex* 3, *P.* 64] ¶ Similarly it can be proved that OB and OC are respectively double of the perpendiculars drawn from S on AC and AB.



11 Let ABC be a triangle and O its ortho-centre. Join AO, BO and CO. Find P, Q, R, and S the centres of the circles circumscribed about the triangles BOC, COA, AOB and ABC respectively (*Prob.* 25). Join PQ, QR and RP. *Then the △ PQR shall be equal in all respects to the △ ABC*

Join AS, BS, CS, BP, CP, AQ and CQ. Then these st lines being the radii of the circles circumscribed about the △s BOC, COA, AOB and ABC, are all equal to one another. [*Ex* 5.]

∴ The figures $BPCS$ and $AQCS$ are each a rhombus

∴ AS is parallel and equal to CQ and BS is parallel and equal to CP

∴ The $\angle ASB$ = the $\angle QCP$.

Now in the triangles ABS and PQC, because $AS = CQ$, $BS = PC$ and the $\angle ASB$ = the $\angle QCP$, therefore the triangles are identically equal (*Th* 4), and therefore $AB = PQ$.

Similarly it may be proved that $QR = BC$ and $PR = AC$.

∴ The triangles ABC and PQR have three sides of the one equal to the three sides of the other.

∴ The △ PQR is equal in all respects to the △ ABC.

12 See fig. Ex 9

It is required to construct a triangle, having given the vertex, the orthocentre, and the centre of the circum-circle.

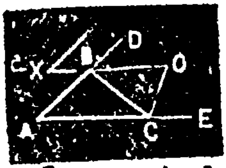
Analysis—Let ABC be the required triangle of which A is the given vertex, O the orthocentre, and S the circum-centre.

The triangle can be constructed if the base BC is known. Let us consider SM to be the perpendicular dropped from S on BC. Then SM is parallel to AO and half of it (*Ex* 10). Hence we have the following construction —

Construction—Join AO and AS. With centre S and radius AS describe a circle. From S draw SM parallel to AO

making $SM = \frac{1}{2} AO$. Through M draw BMC perpendicular to SM cutting the circle at the pts. B and C. Join AB, AC. Then ABC is the required triangle.

Exercises on Loci, P. 211.



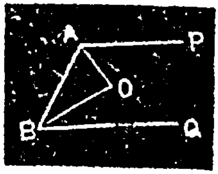
1. Let BC be the given base, and X the given angle, and let BAC be any triangle on the base BC having its vertical $\angle A$ = the $\angle X$. Produce AB, AC to any point D and E, and bisect the ext $\angle$s CBD, BCE by the st. lines intersecting at the ex-centre O opposite to A. *It is required to find the locus of O*

Because the $\angle BOC = 90° - \frac{a}{2}$ (*Ex* 7, *P.* 47),

and the $\angle A$ is constant (*being always equal to the* $\angle X$.)

∴ The $\angle BOC$ is also constant.

∴ The locus of O is the arc of a segment on the fixed chord BC and containing an angle $= 90° \frac{a}{2}$



2. Let AB be any given straight line, and AP, BQ any two parallel st lines drawn from A and B. Bisect the angles PAB, QBA by the st. lines intersecting at O. *It is required to find the locus of O.*

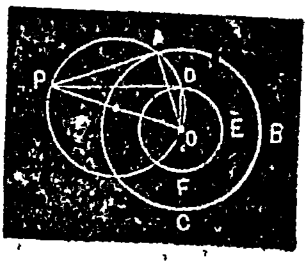
Because the $\angle PAB$ + the $\angle QBA = 180°$ (*Th.* 14).

∴ $\frac{1}{2}$ the $\angle PAB + \frac{1}{2}$ the $\angle QBA = 90°$, i. e. the $\angle OAB$ + the $\angle OBA = 90°$.

∴ The $\angle AOB = 90°$. (*Th* 16, *Inf.* 4)

∴ The locus of O is the circle described on AB as diameter. (*Th* 41).

3. See Ex. 6, P. 165.

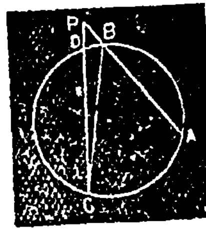


4. Let ABC, DEF be any two of a system of concentric circles whose common centre is O Let P be a fixed pt. from which PA, PD etc. tangents are drawn to these circles. *It is required to find the locus of the points of contact of these tangents,*

Join OA, OD. Then since O and P are fixed points therefore OP is a fixed straight line

And the angles PAO, PDO etc are all right angles (*Th* 46)

∴ The locus of the required points is a circle described upon the diameter OP (*Ex* 1, *P*. 165).



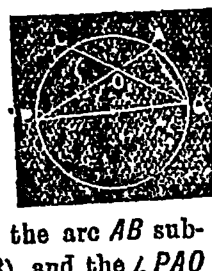
5. Let $ABDC$ be a given circle and B and D two fixed points on it. Let BP DP be drawn any two such straight lines from B and D that they intercept on the circumference an arc AC of constant length *It is required to find the locus of P.*

Since the arcs AC, BD are of constant length

∴ The ∠s DCB and ABC which these arcs respectively subtend at the circumference, are of constant magnitude

Again since the ∠ CPB = the ∠ ABC – the ∠ DCB (*Th* 16) therefore the ∠ CPB, or DPB is also constant

∴ The locus of P is the arc of a segment on the fixed chord BD, and containing an angle = the ∠ ABC – the ∠ PCB

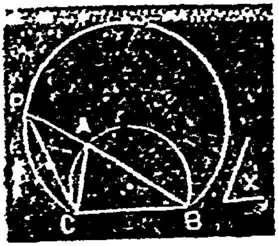


6. Let $ABPQ$ be a circle of which PQ is a diameter Let A, B be two fixed points on its circumference, and let AP and QB intersect at O *It is required to find the locus of O*

A, B being fixed points, the ∠ AQB which the arc AB subtends at the circumference is constant (*Th* 43), and the ∠ PAQ is a right angle (*Th* 41).

∴ The ∠ AOB which is = the ∠ OAQ + the ∠ AQO (*Th* 16), is also constant.

∴ The locus of O is the arc of a segment on the fixed chord AB, and containing an angle = 90° + the ∠ AQO.



7. Let ABC be a triangle described upon the base BC and having its vertical angle equal to the given $\angle X$, and let BA be produced to P such that $BP=BA+AC$. *It is required to find the locus of P.*

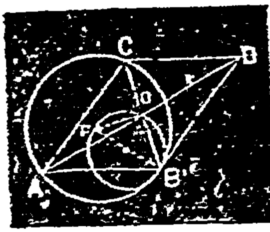
Join PC. Since $BP=BA+AC$ therefore $AP=AC$, and therefore the $\angle APC=$ the $\angle ACP$.

Because the $\angle BAC=$the $\angle APC+$the $\angle ACP$ (*Th* 16).$=$ 2 the $\angle APC$.

$\therefore$ The $\angle APC=\frac{1}{2}$ the $\angle BAC$. But the $\angle BAC$ being equal to the given $\angle X$, is constant.

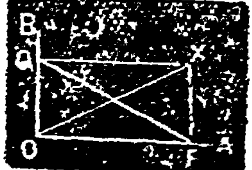
$\therefore$ The $\angle APC$ is also constant.

$\therefore$ The locus of P is the arc of a segment on the fixed chord BC, and containing an angle$=\frac{1}{2}$ the $\angle Y$.



8. Let ABC be a given circle of which AB is a fixed chord. Draw any other chord AC from A, and complete the parallelogram $ACDB$. Draw the diagonals AD, CB intersecting at O *It is required to find the locus of O.*

Since the diagonals of a parallelogram bisect each other therefore O is the middle point of the chord BC : and since this chord passes through the fixed point B, therefore the locus of its middle point O is the circle POB whose diameter $PB=$the radius of the given circle ACB (*Ex.* 3.)



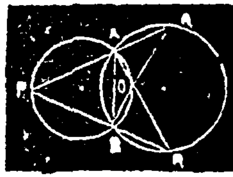
9. Let OA, OB be two rulers placed at right angles to one another, and let PQ be a position of the straight rod PQ which slides between them. From P and Q draw PX, QX perps. to OB and OA intersecting at X. *It is required to find the locus of X.*

Because the figure $OPXQ$ is evidently a rectangle, therefore its diagonals OX and PQ are equal.

Since the rod PQ is of constant length, therefore OX is also of constant length, and the point O is a fixed point.

∴ The locus of X is a circle whose centre is O, and radius=the length of the rod.

10 Let P be a point on one of the two circles intersecting at A and B and let the st lines PA, PB meet the other circles at Q and R Let AR and BQ intersect at O *then it is required to find the locus of O*

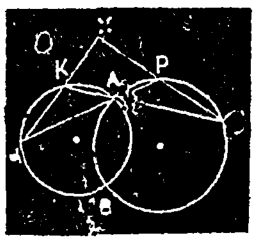


Because A and B are fixed points, therefore the ∠s APB, AQB and ARB are of constant magnitude.

∴ The ∠QBR being equal the ∠QPB+the ∠PQB is constant, and therefore the ∠AOB being equal to the ∠QBR+the ∠ARB is also constant.

∴ The locus of O is an arc of a segment on the fixed chord AB

11. Let AHB and AKB be any two circles intersecting at A and B, and let HAK be a fixed straight line drawn through A and terminated by the circumferences. Also let PAQ, be any other straight line similarly drawn. Join HP and QK and produce them to intersect at X *It is required to find the locus of X.*



Since the ∠HPQ=the ∠PXQ+the ∠PQX (Th 16).

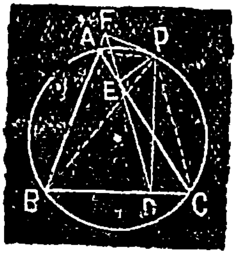
∴ The ∠PXQ=the ∠HPQ-the ∠PQX.

Since H, A and K are fixed points therefore the angles HPA and AQK which the arcs HK and AK subtend at the circumferences are of constant magnitude.

∴ Their difference is also constant, that is, the $\angle PXQ$ is constant.

∴ The locus of X is an arc of a segment on the fixed chord HAK, and containing an angle = the $\angle HPQ$ – the $\angle PQX$

Exercises on Simson's Line P 212.



1. Let P be any point on the circumcircle of the △ ABC. and let PD, PF be the perps. drawn from P on BC and AB respectively. Join FD and let it cut AC at E. Join PE.

Then shall PE be perpendicular to AC.

Join AP, BP and CP. Because the ∠s BFP and BDP are rt. angles, therefore the points P, F, B and D are concyclic (*Converse of Th* 40), and therefore the $\angle FPB$ = the $\angle FDB$ [*Th* 39].

Also the $\angle APB$ = the $\angle ACB$ (*Th. 39*).

∴ The $\angle FPA$ = the $\angle FPB$ – the $\angle APB$ (*Th* 16).
= the $\angle FDB$ – the $\angle ACB$
= the $\angle DEC$ (*Th.* 16).
= the $\angle AEF$ (*Th.* 3).

∴ The points A, E, P and F are concyclic (*Converse of Th.* 39).

∴ The ∠s AFP and AEP are supplementary (*Th.* 40).

But the $\angle AFP$ is a rt. angle, therefore the $\angle AEP$ is also a rt. angle.

∴ PE is perpendicular to AC.

2. See fig Ex. 1.

Let P be any such point that D, E and F, the feet of the perpendiculars drawn from it on the side of the given triangle ABC, are collinear. *It is required to find the locus of P*

Because the ∠s PEA and PFA are rt. angles, therefore the pts. P, F, A and E are concyclic, and therefore the $\angle AFE$ = the $\angle APE$ (*Th.* 39).

Again because the ∠s PEC, PDC are rt. angles, therefore

the pts P, E, D and C are concyclic ∴ The $\angle DPC =$ the $\angle DEC$ (*Th.* 39).

$=$ the $\angle AEF$

$=$ the $\angle APE$

To each of these equals add the $\angle APD$, then the $\angle APC =$ the $\angle FPD$.

Again because the $\angle$s BDP, BFP are rt angles therefore the pts B, D, P and F are concyclic, and therefore the $\angle FPD$ and FBD are supplementary (*Th.* 40). But the $\angle FPD$ has been proved to be equal to the $\angle APC$

∴ The $\angle$s ABC and APC are also supplementary

∴ The pts A, B, C, and P are concyclic [*Converse of Th* 40].

∴ The locus of P is the circumcircle of the $\triangle$ ABC

3. Let $ABC, AB'C'$ be any two triangles having the $\angle A$ common to both. Let the circum-circles of these triangles meet again at P. From P draw PD, PE, PF and PG perpendiculars on BC, CA, AB and $B'C'$ respectively. *It is required to prove that the pts. D, E, F, and G are collinear*

Because PD, PE and PF are perpendiculars from P on the sides of the $\triangle ABC$ therefore the pts D, E and F are collinear [*Prop* V *P* 212].

Again because PG, PE and PF are perpendiculars from P on the sides of the $\triangle$ $AB'C'$, therefore the pts G, E and F are collinear (*Prop* V. *P* 212)

∴ The pts D, E, F and G are collinear.

4. Let ABC be a triangle inscribed in a given circle, and let P be any point on this circle. Let O be the ortho-centre of the $\triangle$ ABC. Join OP. From P draw PD, PE, and PF perps. to BC, CA and AB respectively. *It is required to prove that OP is bisected by the st. line DEF.*

Let FP meet the circle again at G. Join GC, and produce FG to H making $PH=FG$

Find Q the circum-centre of the $\triangle ABC$ [*Prob.* 25] and draw QK, QL perps. to AB and PG respectively. Then K and L are the middle points of AB and PG [*Th* 31].

Join OC, then $OC=2QK$ [*Ex* 10, P 209] $=2FL=FH$.

Because the angles AEP and AFP are rt. angles therefore the points A,E,P and F are concyclic, and therefore the $\angle PAE$ =the $\angle PFE$ [*Th* 39].

Again because the pts. A, C, P and G are concyclic, ∴ the $\angle PGC$=the $\angle PAC$, or PAE.

∴ $\angle PGC=\angle PFE$, and therefore GC is parl. to FD [*Th* 13].

If CO intersect DF at N, then $FNCG$ is a parallelogram, and therefore $CN=FG=PH$

And since $OC=FH$ therefore $ON=FP$.

Also ON is parallel to FP, for each is perp to BF.

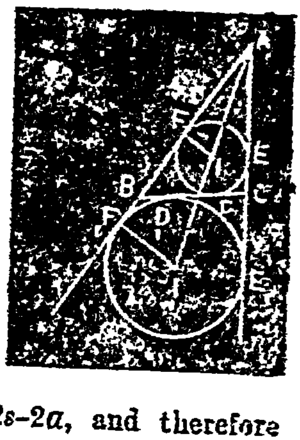
∴ The fig $FONP$ is a parallelogram.

Hence the diagonals FN, OP bisect each other at R.

∴ OP is bisected by the st line DEF.

Proof of the equalities on Prop. VI, P. 213.

(i) Because AE, AF are two tangents from A to the inscribed circle hence $AE=AF$ [*Th* 47 *Cor.*]



Similarly it may be proved that $BD=BF$, and $CD=CE$

Now $AB+BC+CA=(AF+AE)+(BF+BD)+(CE+CD)$.

$$=2AE+2BD+2CD$$

$$=2AE+2BC$$

That is $2s=2AE+2a$, ∴ $2AE=2s-2a$, and therefore $AE=s-a$

Similarly it can be proved that $BD=BF=s-b$, and $CD=CE=s-c$.

(ii) Because AE_1 AF_1, are two tangents from A to the escribed circle, therefore $AE_1 = AF_1$ [*Th.* 47 *Cor*].

Similarly $BF_1 = BD_1$, and $CE_1 = CD_1$

$\therefore AB+BC+CA = AB+BD_1+AC+CD_1 = AB+BF_1+AC+CE_1 = AF_1+AE_1 = 2\ AE_1$

i. e $2s = 2\ AE_1$. Therefore $AE_1 = AF_1 = s$

(iii) Because $CE_1 = AE_1 - AC = s-b$, ¶ $\therefore CD_1 = CE_1 = s-b$.

And because $BF_1 = AF_1 - AB = s-c$, ¶ $\therefore BD_1 = BF_1 = s-c$

(iv) Because $CD = s-c$, also $BD_1 = s-c$ $\therefore BD_1 = CD$.

Again because $BD = s-b$, also $CD_1 = s-b$ ¶ $\therefore BD = CD_1$

(v) $EE_1 = AE_1 - AE = s-(s-a) = a$

and $FF_1 = AF_1 - AF = s-(s-a) = a$

$\therefore EE_1 = FF_1 = a$

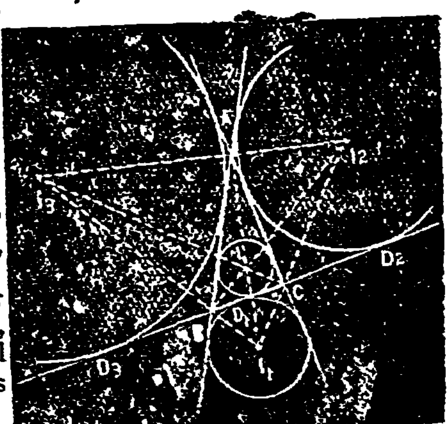
(vi) Area of the $\triangle ABC = \frac{1}{2}(a+b+c)\,r$ [*Ex* 5 P. 198] $= rs$.

Also . ... $= \frac{1}{2}(b+c-a)\,r_1$ [*Ex.* 6 P. 198] $= r_1(s-a)$, because $(s-a) = \frac{1}{2}(a+b+c) - a = \frac{1}{2}(b+c-a)$

(vii) If the $\angle C$ be a rt. angle, then the figures $IDCE$, and $I_1 D_1 CE_1$ would be rectangles

$\therefore r = ID = CE = (s-c)$ and $r_1 = I_1D_1 = CE_1 = (s-b)$

Proof of the properties on Prop. VII P, 214



(i) Because IA bisects the $\angle BAC$ (*Prob.* 26), and I_1A also bisects the $\angle BAC$ (*Prob* 27), therefore the pts A, I and I_1 are collinear

Similarly it can be proved that the pts. B, I and I_2 as also the pts C, I and I_3 are collinear

(ii) Because AI_1, AI_2 are the internal and external bisectors

of the $\angle A$, therefore the $\angle I_1 A I_2$ is a right angle. Similarly the $\angle I_1 A I_3$ is a right angle. Hence the pts. I_2, A and I_3 are collinear.

Similarly it can be proved that the pts. I_3, B and I_1 as also the pts. I_1, C and I_2 are collinear.

(iii) Because AI_1 and AI_2 bisect the *internal and external* $\angle A$ therefore $I_1 A$ is perp. to $I_2 I_3$ ¶ Similarly it can be proved that $I_2 C$ is perp. to $I_1 I_3$, and that $I_3 B$ is perp. to $I_1 I_2$ ¶ $\therefore ABC$ is the pedal triangle of the $\triangle I_1 I_2 I_3$.

$\therefore$ The $\triangle s$ BI_1C, CI_2A, AI_3B are equiangular to one another and to the $\triangle I_1 I_2 I_3$ [*Prop.* II *Cor.* (E). *P.* 209].

(iv) If the inscribed circle touches the sides BC, CA and AB at D, E and F, then the $\angle FDE = 90° - \frac{A}{2}$ (*Ex.* 5, *P.* 206)

Also the $\angle BI_1C = 90° - \frac{A}{2}$ (*Ex.* 7, *P.* 47).

$\therefore$ The $\angle FDE$ = the $\angle BI_1C$. Similarly it may be proved that the $\angle DEF$ = the $\angle AI_2C$, and that the $\angle EFD$ = the $\angle AI_3B$ ¶ $\therefore$ The $\triangle s$ DEF and $I_1 I_2 I_3$ are equiangular.

(v) Because I is the orthocentre of the $\triangle I_1 I_2 I_3$ [*Proved in case* III].

$\therefore$ Of the four pts. I, I_1, I_2 and I_3 each is the orthocentre of the triangle whose vertices are the other three (*Ex.* 4, *P.* 203)

(vi) Since I is the orthocentre of the $\triangle I_1 I_2 I_3$ therefore the three circles which pass through two vertices of the $\triangle I_1 I_2 I_3$ and I, are each equal to the circum-circle of $\triangle I_1 I_2 I_3$ [*Ex.* 3, *P.* 209].

$\therefore$ The four circles, each of which passes through three of the points I, I_1, I_2 and I_3 are all equal.

Exercises on the Triangles and its circles, P. 215.

1. See fig. Last Exercise.

(i) $DD_1 = BD_1 - BD = s - (s - b) = b$ (*Ex.* (I) & (ii), *P.* 213).

and $D_1H_3 = CD_3 - CD_1 = s-(s-b) = b$ [*Exs* (II) & (III), *P* 213]

∴ $DD_2 = D_1D_3 = b$

(ii) $DD_3 = CD_3 - CD = s-(s-c) = c$ [*Exs.* (i) & (ii), *P.* 213]

and $D_1D_2 = BD_2 - BD_1 = s-(s-c) = c$ (Exs. (ii) & (iii), *P.* 213)

∴ $DD_3 = D_1D_2 = c$.

(iii) $D_2D_3 = D_1D_3 + D_1D_2 = b + c$.

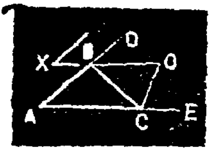
(iv) $DD_1 = DD_2 \sim D_1D_2 = b \sim c$.

2. See fig Last Exercise.

Because I is the orthocentre of the $\triangle\ I_1I_2I_3$ and ABC is its pedal triangle, also I, I_1, I_2 and I_3 are the centres of the inscribed and escribed circles of the $\triangle\ ABC$.

∴ The orthocentre and vertices of a triangle are the centres of the inscribed and escribed circles of the pedal triangle.

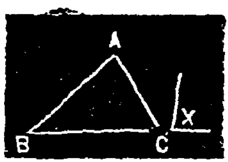
3 Let BC be the given base, X the given angle, and let ABC be any triangle on the given base BC having the vertical $\angle A$ equal to the $\angle X$ Produce AB, AC to any points D and E, and bisect the $\angle s\ CBD$, BCE by the st lines BO CO intersecting at O *It is required to find the locus of O*



Because the $\angle BOC = 90° - \frac{A}{2}$ [*Ex*, 7, *P* 47], and the $\angle A$ is constant, therefore the $\angle BOC$ is also constant

∴ The locus of O is the arc of a segment on the fixed chord BC containing an angle $= 90° - \frac{A}{2}$

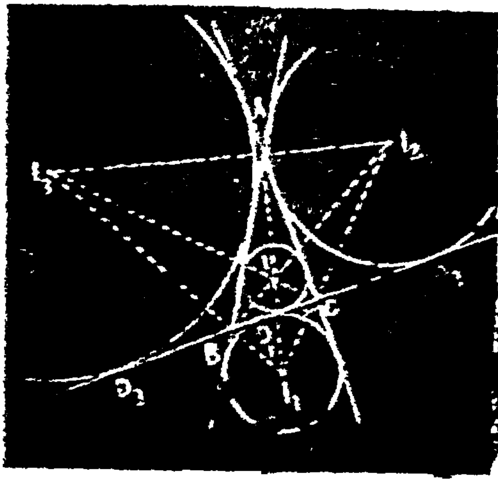
4 Let BC be the given base, and X the given angle, and let ABC be any triangle on the given base BC, having the vertical $\angle A$ equal to the $\angle X$ *It is required to prove that the circum-centre of the $\triangle\ ABC$ is fixed*



Because the $\angle A$ is constant, and the base BC is fixed therefore the locus of A is the arc of a segment on the

fixed chord BC, containing an angle equal to the $\angle X$. But this is the segment of the circum-circle of the triangle BAC, therefore the circum-circle is fixed, and hence its centre is also fixed.

5. Let ABC be any triangle on the given base BC, and having the vertical $\angle BAC=$ the given vertical angle. Let I_2 be the centre of the escribed circle touching the side AC. *It is required to find the locus of I_2.*

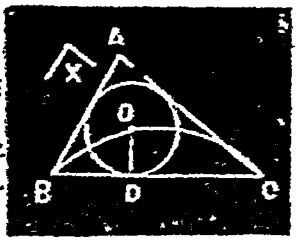


Because the $\angle s$ I_1BI_2 and I_1AI_2 are rt. angles, therefore the pts. I_1, B, A and I_2 are concyclic, and therefore the

$\angle BI_2I_1=$ the $\angle BAI_1$ in the same segment ($Th.$ 33) $=\frac{A}{2}$

$\therefore$ The locus of I_2 is the arc of a segment on the fixed chord BC containing an angle $=\frac{A}{2}$

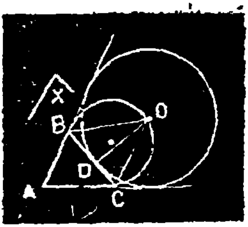
6. Let BC be the given base, X the given angle. and D the point of contact with the base BC of the in-circle. *It is required to construct the triangle.*



Locus of O is the arc of a segment on the fixed chord BC, and containing an angle $=90^\circ+\frac{X}{2}$ (*Prop. IV., P. 210*).

From D draw DO perp. to BC meeting this arc at O. Then O is the in-centre of the triangle, and OD the in-radius.

With centre O and radius OD draw a circle. From B and C draw tangents to this circle, intersecting at A. Then ABC is the required triangle.

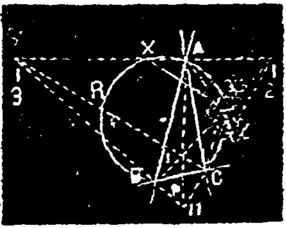


7. Let BC be the given base, X the given vertical angle, and D the point of contact with the base BC of the escribed circle. *It is required to construct the triangle.*

Locus of O is the arc of a segment on the fixed chord BC containing an angle $=90^\circ-\frac{x}{2}$ [*Ex.* 1, *P.* 211].

From D draw DO perp. to BC meeting this arc at O. Then O is the centre, and OD the radius of the escribed circle.

With centre O and radius OD draw a circle. From B and C draw tangents to this circle, and let them intersect at A. Then ABC is the required triangle.



8. Let I be the centre of the inscribed circle, and I_1, I_2, I_3 the centres of the escribed circles, of the $\triangle ABC$, and let the circum-circle of the triangle ABC cut $II_1 II_2 II_3$ at the pts P, Q and R respectively.

Then II_1, II_2, and II_3 shall be bisected at the pts. P. Q and R.

Join AQ, CQ. Then because the $\angle AQC=180^\circ-B$ (*Th* 40). and the $\angle AI_2C=90^\circ-\frac{B}{2}$(*Ex* 7. P. 47), therefore the $\angle AQC=$ 2 the $\angle AI_2C$.

Again because the $\angle$s IAI_2 and ICI_2, are rt. angles therefore the circles on diameter II_2 passes through A and C (*Ex* 1 P. 165); and because the $\angle AQC=2$ the $\angle AI_2C$, therefore Q is the centre of this circle ¶ Hence II_2 is bisected at Q.

Similarly it can be proved that II_1 is bisected at P, and II_3 is bisected at R.

9. See fig. Ex. 8

Let I_2, I_3 be the centres of the escribed circles which touch the sides AC and AB of the $\triangle ABC$ respectively. *It is required to prove that the pts. B, C, I_2 and I_3 shall lie on a circle whose centre lies on the circum-circle of the $\triangle ABC$.*

Because the $\angle$s I_2BI_3 and I_2CI_3 are rt. angles, therefore the pts. I_2 C, B and I_3 lie on a circle whose diameter is I_2I_3. ¶ Bisect I_2I_3 at X, then X is the centre of this circle.

Join QX, then because II_3 is bisected at Q (*Ex.* 8 and I_2I_3 at X, therefore QX is parl. to II_2 (*Ex.* 2, *P.* 64), and therefore the ext. $\angle AXQ$ = the int. $\angle I_2I_3C$ (*Th.* 14).

Again because the $\angle$s IAI_3 and IBI_3 are rt angles, therefore the pts. I, A, I_3 and B are concyclic, and therefore the $\angle AI_3I$=the $\angle ABI$ [*Th.* 39].

∴ The $\angle AXQ$=the $\angle ABI$ or the $\angle ABQ$ and therefore the points A, X, B, Q are concyclic.

But the pt. Q lies on the circum-circle of the $\angle ABC$. therefore the pt. X also lies on the circum-circle of the $\triangle ABC$.

10. See fig. Ex. 1, P 213.

Let A, B, and C be any three given points *It is required to to draw with A, B, and C, as centres, three circles which may to uch one another two by two.*

(i) Let the inscribed circle of the $\triangle ABC$ touch the sides BC, CA and AB at the points D, E and F respectively.

Then because $AE=AF$, $BD=BF$, and $CD=CE$ (*Ex.*1, *P.*213).

∴ The circles described with centres A, B and C and radii AE, CD and BF will touch each other *externally* two by two.

Note—*See also Ex.* 12, *P* 206.

(ii) Let the escribed circle whose centre is I_1 touch the sides BC, CA and AB at the points D_1, E_1 and F_1 respectively. Then because $AE_1=AF_1$, $BD_1=BF_1$ and $CD_1=CE_1$

(*Exs* (ii) and (iii), *P* 213] hence the circles described with centres A, B and C and radii AE_1, CD_1 and BF_1 will touch each other two by two

Similarly if the escribed circles whose centres are I_2 and I_3 touch the sides BC, CA and AB at the points D_2, E_2, F_2 and D_3, E_3, F_3 respectively, then it can be shown that the circles described with centres A, B, C and radii AE_2, CD_2 and BF_2, as also the circles described with centres A, B, C, and radii AE_3, CD_3 and BF_3 will touch each other two by two

Thus we see that there are *four* solutions of this problem.

11. See fig. Ex. 5.

Let I_1, I_2, I_3, be the given centres of the three escribed circles *It is required to construct the triangle.*

Analysis Let ABC be such a triangle. Join $I_1 I_2$, I_3 and $I_3 I_1$ and from I_1, I_2 I_3 draw $I_1 A$, $I_2 B$ and $I_3 C$ perps to the opposite sides, and let them intersect at the pt I.

Then because I is the orthocentre of the $\triangle$ $I_1I_2I_3$, and the pts. A, I, I_1 are collinear; so also the pts B, I, I_2 and C, I, I_3 are collinear [*Exs* (iv) & (i) *P.* 214]

$\therefore$ ABC is the pedal triangle of the $\triangle$ $I_1I_2I_3$. Hence we have the following construction —

Cons. Join I_1I_2, I_2I_3 and I_1I_3, and from I_1, I_2 and I_3 draw I_1A I_2B and I_3C perps to the opposite sides. Join AB, BC and CA Then ABC is the required triangle.

12. See fig Ex 5

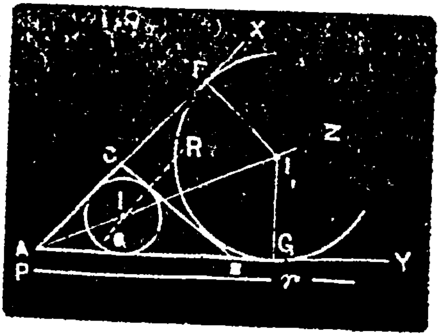
Let I be the centre of the inscribed circle, and I_1, I_2 the centres of two escribed circles. *It is required to construct the triangle*

Analysis—Let ABC be such a triangle. From I_1, I_2 draw I_1B and I_2A perps. to I_2I and I_1I produced, and from I draw IC perp to I_1I_2.

Then because the $\angle s\ I_1AI_2$ and I_1BI_3 are rt. angles, therefore a circle described on I_1I_2 as diameter will pass through B and A (*Ex.* 1, P. 165). Hence we have the following construction :-

Cons.—Upon the diameter I_1I_2 draw a semi-circle. Join I_1I and I_2I and produce them to meet the circumference at pts. A and B. From I draw IC perp. to I_1I_3. Join AB, BC and CA. Then ABC is the required triangle.

13 Let XAY be the given vertical angle, P the semi-perimeter, and r the radius of the inscribed circle *It is required to construct the triangle.*



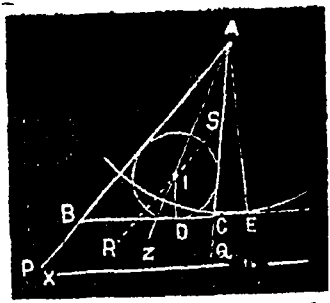
Analysis—Let ABC be such a triangle, and I, I_1 be the centres of the inscribed and the escribed circles touching the side BC. Then the pts A, I and I_1 are collinear. Through I draw a st. line QIR parl to AX, then its distance from $AX=r$ From I_1 draw I_1F/G perps to AX and AY; then $AF=AG=P$ (*Ex* (ii), P. 213) Also BC is the common tangent to the inscribed and the escribed circle

Hence we have the following construction :—

Cons—Draw a st line QR parl to AX, and at a distance of r from it From AX and AY cut off AF and AG each equal to P At the pts F and G draw perps. to AX and AY intersecting at I_1 With centre I_1 and radius I_1F or I_1G, draw a circle.

Join AI_1 and let it intersect QR at I. With centre I and radius$=r$ draw another circle

Draw a common tangent to these two circles intersecting AX and AY at the pts B and C respectively. Then ABC is the required triangle.



14. Let PAQ be the given vertical angle. X the length of the perpendicular from the vertex to the base, and r the radius of the inscribed circle. *It is required to construct the triangle.*

Analysis—Let ABC be such a triangle, and let I be the centre of the inscribed circle. Join AI, then AI bisects the $\angle PAQ$. Through I draw a st. line $R.S$ parl. to AP, then its distance from $AP = r$ From A draw AE perp. to C, then $AE = X$ From I draw ID perp to BC

With centres I and A and radii $= r$ and X respectively, draw two circles Then because the $\angle s$ at D and E are rt. angles, therefore BC is the common tangent to these two circles.

Hence we have the following construction :—

Cons—Bisect the $\angle\ PAQ$ by the st line AZ, and draw a st. line RS parl. to AP and at a distance of r from it Let AZ intersect RS at the point I

With centres I and A, and radii$=r$ and X respectively draw two circles Draw a common tangent to these two circles, and let it intersect AP and AQ at the pts. B and C respectively. Then ABC is the required triangle

15 See fig Ex 8.

Let ABC be a triangle, and I the centre of the inscribed circle *It is required to prove that the centres of the circles circumscribed about the triangles BIC, CIA and AIB lie on the circumference of the circle circumscribed about the triangle ABC.*

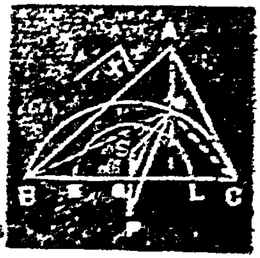
Let I_1, I_2, I_3 be the centres of the three escribed circles Join AI_1, BI_2 and CI_3, then each of them passes through I. Join I_1I_2, I_2I_3 and I_3I_1, and let the circle circumscribed about the triangle ABC, cut II_1, II_2 and II_3 at the pts P, Q and R respectively Join AQ, CQ

It has been proved in Ex. 8, that pts. A, I, C, I_2 lie on a circle. and that Q is its centre. Hence the centre Q of the circle circumscribed about the triangle CIA lies on the circum-circle of the triangle ABC.

Similarly it can be shown that P and R are the centres of the circles circumscribed about the triangles BIC and AIB and that they lie on the circum-circle of the triangle ABC.

Exercises on the Nine-Points Circle, P, 218.

1. Let BC be the given base, and X the given vertical angle, and let ABC be a triangle on the base BC having the vertical $\angle BAC$ = the $\angle X$. *It is required to find the locus of the centre of the nine-points circle.*



Because the base and the vertical angle are given therefore the circum-centre S is a fixed point (*Ex 4, P. 215*) and the locus of the orthocentre O is an arc of a segment on the fixed chord BC, whose centre is P, and which contains an angle $=180°-A$ (*Prob, III, P. 210*).

Again because N, the centre of the nine-points circle is the middle point of OS, and S is a fixed point, also O moves on an arc of a circle; therefore the locus of N is an arc of a segment whose centre Q is the middle point of SP, and whose radius $QN=\frac{1}{2}OP$ (*Ex. 10, P. 94*).

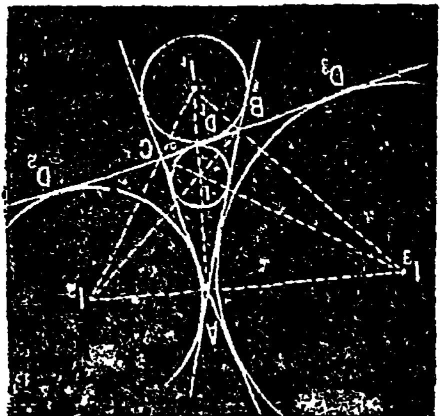
2. See fig. Prop. VIII, P. 216.

Let ABC be a triangle whose orthocentre is O. *It is required to prove that the nine-points circle of the triangle ABC is also the nine-points circle of each of the triangles AOB, BOC and COA.*

Because the nine-points circle of the $\triangle ABC$ passes through the middle points of AB, AO and OB, therefore it passes through the middle points of the sides of the $\triangle AOB$. Hence it is also the nine points circle of the $\triangle AOB$.

Similarly it can be proved that it is also the nine-points circle of the $\triangle$s BOC and COA

3 Let I, I_1, I_2 and I_3 be the centres of the inscribed and escribed circles of a triangle ABC. *It is required to prove that the circle circumscribed about the $\triangle$ ABC is the nine-points circle of each of the four triangles formed by joining three of the points I, I_1, I_2 and I_3.*



Because $I_1 A$, $I_2 B$ and $I_3 C$ are perps to I_2I_3, I_1I_3 and I_1I_2 respectively, therefor the pt I is the orthocentre and ABC is the pedal triangle of the $\triangle$ $I_1I_2I_3$.

∴ The circum-circle of the $\triangle ABC$ is the nine-points circle of the $\triangle I_1I_2I_3$.

∴ It is also the nine points circle of each of the triangles II_1I_2, II_1I_3 and $II_2 I_3$ (*Ex.* 2)

4. *It is required to prove that all triangles which have the same orthocentre and the same circumscribed circle, have also the same nine-points circle.*

Because the triangles have the same circumscribed circle, hence their common circum-centre is a fixed point and the circum-radius is of constant length.

Again because the triangles have the same ortho-centre, hence it is also a fixed point.

∴ The centre of the nine-points circle, which is the middle point of the st. line joining the circum-centre and the orthocentre, is also a fixed point

Again, because the circum-radius is of constant length, hence the radius of the nine-points circle, which is equal to half of the circum-radius, is also of constant length.

∴ All the triangles have same nine-points circle.

5. See fig Ex. 3

Let $I_1 I_3$ be the given base, and $I_1 I_3 I_2$ the given vertical angles. *It is required to show that one side and one angle of the pedal triangle ABC, are constant.*

Because the $\angle$s IBI_1 and ICI_1 are rt. angles, therefore the pts. I, B, I_1, C are concyclic [*Converse of Th.* 40].

∴ The $\angle BCI$ = the $\angle BI_1 I$ (*Th* 39).

$= 90°$ – the $\angle I_1 I_3 A$ (because $I_1 A I_3$ is a rt angle).

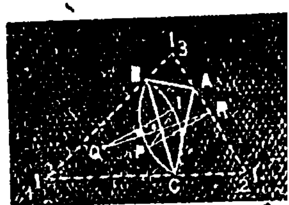
But the $\angle I_1 I_3 A$ is given constant, therefore the $\angle BCI$ is also constant. Hence the $\angle BCA$, which is equal to twice the $\angle BCI$, is also constant.

Again because the base and the vertical angle are given therefore the centre of the circumscribed circle is fi ed (*Ex.* 4. *P.* 215), and the radius of the circumscribed circle is of constant length.

Hence the radius of the nine-points circle, which is equal to half of the circum-radius, is also of constant length, *i. e.* the radius of the circum-circle of the $\triangle ABC$ is constant.

∴ AB the chord of the circum-circle of the $\triangle ABC$ subtended by a constant angle BCA is of constant length.

∴ One side AB and one angle BCA of the pedal $\triangle ABC$ are given.



6. Let ABC be a triangle on the given base BC and having the given vertical angle BAC. Let $I_1 I_2$ and I_3 be the three escribed centres of the $\triangle ABC$, and let P be the circum-centre of the $\triangle I_1 I_2 I_3$. *It is required to find the locus of P.*

Let I and O be the centres of the inscribed and the circumscribed circles of the $\triangle ABC$. Then because the base BC and the vertical $\angle A$ are given, hence O is a fixed point (*Ex* 4. *P*. 215), and the locus of I is the arc of the segment BIC on the fixed chord BC containing an angle$=90°\frac{\pi}{2}$ (*Prop*) IV, P 210)

Because I is the orthocentre of the $\triangle$ $I_1I_2I_2$ and ABC is its pedal triangle, therefore the circum-circle of the $\triangle$ ABC is the nine-points circle of the $\triangle$ $I_1I_2I_3$ (*Obs Prop*. VIII, *P*. 216), and therefore O is the centre of the nine-points circle of the $\triangle$ $I_1I_2I_3$. Join IP, then IP is bisected at O (*Prop*. (*i*), *P* 217).

Let Q be the centre of the circle of which BIC is an arc, then Q is a fixed point. Join QO, and produce it to R such that $OR=OQ$, then R is also a fixed point

Join IQ and PR. Then in the $\triangle$s IOQ and POR, because $IO=OP$, $OQ=OR$ and the $\angle IOQ=$the $\angle POR$

$\therefore$ The triangles are identically equal *(Th. 4)*, and therefore $PR=IQ$.

But IQ being the radius of the circle of which BIC is an arc is of constant length.

$\therefore$ PR is also of constant length.

$\therefore$ the locus of P is an arc BPC on the fixed chord BC, the centre of which is R, and radius $PR=IQ$.

IMPORTANT NOTICE.

We have also prepared chorts containing o... *eleven* formulæ which will enable the School-Leav-ing Students to solve all questions in Mensuratio[n] for finding the areas of all the important regula[r] figures and the radii of their inscribed and circums-cribed circles. These choits are offered at a ver[y] moderate price of one anna and a half each.

N. B.—Books bearing no signature o[f] the author are to be supposed forge[d]

Kanhaiya Lal Sharma

Kanhaiya lal Sharma.